Constructions for Higher Dimensional Perfect Multifactors

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Abstract

Perfect maps, factors and multifactors can be viewed as higher dimensional analogues of de Bruijn cycles and factored versions of these cycles. We present a unified framework for two basic techniques, concatenation and integration (also called the inverse of Lempel's homomorphism), used to construct perfect multifactors. This framework simplifies proofs of known results and allows for extension of the basic constructions. In particular, we give the first general results on the inverse of Lempel's homomorphism in dimensions three and higher.

1 Introduction

What has come to be known as a de Bruijn cycle (see [4] for more history) is a periodic k-ary string in which every k-ary substring of a given size appears exactly once (periodically). For example, in the period nine string

$$001121022|001121022|0011\ldots$$

each ternary string of length two appears exactly once with period nine. We will usually represent such a string with a fundamental block, writing 001121022 with the periodicity understood. The position of a substring is its location in the block, starting with position 0. So in the example above 00 appears in position 0, 21 in position 4 and 20 in position 8.

There has been some recent interest in higher dimensional analogues of de Bruijn cycles. These have been called de Bruijn tori and perfect maps. Viewing

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{matrix}$$

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periodically in both dimensions (toroidally), each binary 2×2 array appears exactly once, for example, $\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}$ in position $(0, 1)$ and $\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}$ in position $(3, 3)$. There are obvious necessary conditions for the existence of such maps and it is conjectured that these are also sufficient. These conditions are noted in Lemma 3 and the conjecture following the lemma.

Although there have been various methods of constructing perfect maps, two techniques have played a central role. The method of *concatenation* was introduced and developed by Ma [10], Cock [1] and Etzion [2] and has been described for all dimensions. The method of *integration* (sometimes called the inverse of Lempel's homomorphism) was introduced in Fan, Fan, Ma and Sui [3]. Methods for integration of perfect factors have been extended and refined by a number of authors, however only in one and two dimensions. In particular, when the entries are from a finite field, Paterson [15], [16] made use of the linear complexity of a sequence to allow repeated applications of integration. With this, he showed that obvious necessary conditions for the existence of 2-dimensional k-ary de Bruijn tori are sufficient when k is a prime power.

Applying the techniques of concatenation and integration requires introduction of two new objects, perfect factors, which generalize perfect maps and a generalization of perfect factors called perfect multifactors.

Perfect factors can be thought of as a factorization of a de Bruijn cycle (or torus) into a collection of smaller cycles (tori). Perfect factors were introduced by Lempel [9]. Two dimensional perfect factors are mentioned in [18] and higher dimensional versions in [6]. Extensive study of one dimensional perfect factors can be found in [2] and [16].

Perfect multifactors are perfect factors in which each substring (subarray) of a given size appears several times, once in each location relative to a given modulus. Perfect multifactors were introduced by Mitchell [11]. Two dimensional perfect multifactors were introduced by Paterson [18]. Perfect multifactors were introduced and have been used to produce other perfect factors over non prime power alphabets. See for example [11], [14], [18], or see [12] for another variation. We will not discuss these applications here but rather discuss the role of perfect multifactors in concatenation and integration.

In order to have enough power to attack problems of constructing perfect maps, we must look at the broader problem of constructing perfect multifactors. Perfect factors in dimension $d-1$ along with one dimensional perfect multifactors are used in concatenation to produce d dimensional perfect maps (and factors). Perfect multifactors in dimension $d - 1$ are a key to applying integration in dimension d.

We now briefly discuss repeated applications of the constructions described in this paper. Repeated applications of integration can be done by switching the direction along which integration occurs as in [5] for the two dimensional case. The extra flexibility in higher dimensions should allow even more with this approach. Repeated applications of integration can also be done using linear complexity of a sequence when the alphabet is a finite field as in [15] and [16]. It seems possible that these approaches for repeated application of integration applied to the new results on integration in higher dimensions could provide a basis for a proof that the obvious necessary conditions (see Lemma 3) for k-ary de Bruijn tori in higher dimensions can be shown to be sufficient when k is a prime power. However for general multifactors and even for de Bruijn tori when the alphabet size k is not a prime power other methods will likely be needed. Indeed, the general cases have not even been settled in 1 and 2 dimensions for non-prime power alphabets.

In this paper we will begin by giving a number of motivating examples in Section 2. In Section 3 we will give more formal definitions and discuss obvious necessary conditions which are believed to be sufficient. In Section 4 we describe general results for the construction methods of concatenation and integration. For concatenation almost all of the cases where our results apply have been mentioned previously in the literature, but they have not all been written down in a unified format. To aid this, we will introduce another class of one-dimensional strings, perfect multifactor pairs, which have implicitly been used in previous works. For integration, what has been missing is a description of this construction in dimensions 3 and higher as well as integration applied to perfect multifactors in two dimensions. Additionally, the role of perfect multifactors in integration has usually not been made explicit. We will do so here. This allows us to state new broad results for integration.

2 Examples

The notation for discussing perfect multifactors can get quite cumbersome. In this section we present a number of examples to illustrate perfect multifactors as well as the methods of integration and concatenation. A more formal presentation will be in Sections 3 and 4. We adopt the notation of [8], informally in this section and formally in the next.

Example 1: Let us begin with a simple example. Recall the de Bruijn cycle

$$001121022$$

from the introduction. It is a 3-ary string of period 9 in which each length 2 substring appears exactly once. We will call this a $(9;2)_3$-dBS (de Bruijn sequence). Consider the two dimensional array

$$\begin{array}{ccccccccc}
0&0&0&1&0&0&0&1&0\\
0&0&1&2&2&1&2&2&1\\
1&1&1&1&2&1&2&1&1\\
1&1&2&0&0&2&0&0&2\\
2&2&1&2&0&1&0&2&1\\
1&1&0&2&1&0&1&2&0\\
0&0&2&0&1&2&1&0&2\\
2&2&2&0&2&2&2&0&2\\
2&2&0&1&1&0&1&1&0
\end{array}$$

Viewing this 9×9 array toroidally, every 2×2 3-ary subarray appears exactly once. This is called a $((9,9);(2,2))_3$-dBT (de Bruijn torus).

The method of construction is the simplest version of concatenation. Each column is a shifted copy of the previous de Bruijn cycle. The shifts follow the pattern 012345678. Column 1 is obtained by shifting column 0 by 0, column 2 is obtained by shifting column 1 by 1, ... The last 8 indicates that shifting column 8 returns us to the start, column 0 so we can view the array periodically (or as a torus). The subarray $\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}$ appears in position $(0, 2)$ since 01 and 12 appear in 001121022 shifted by 2, hence we look for this subarray where a shift of 2 occurs, starting in column 2. Similarly, each subarray can be found and because of the size, each must appear exactly once.

Example 2: Consider the string

000011210220112102201121022

obtained by writing three 0's followed by three copies of the string 01121022 (i.e., the de Bruijn cycle 001121022 with the first 0 deleted). In this string with period 27, every 3-ary substring of length 2 appears exactly 3 times, once in each position modulo 3. We call this a $(27; 2; 1)_3[3]$-PMF (perfect multifactor). Shifting by 3 and by 6 we get two additional strings

022000011210220112102201121 121022000011210220112102201

for a set of 3, period 27 strings in which each length 2 substring appears appears exactly once in each position modulo 9; a $(27; 2; 3)_3[9]$-PMF. These will be the set of 3 starters for integration in our next example.

Example 3: We now use the $(27; 2; 3)_3[9]$-PMF from Example 2 to integrate the $((9, 9); (2, 2))_3$-dBT from Example 1. We illustrate with the second string of the perfect multifactor as a starter. Below the starter we have written 3 copies of the $((9, 9); (2, 2))_3$-dBT. This is our intermediate array in the construction.

$$
\begin{array}{ccccccccc|ccccccccc|ccccccccc}
0&2&2&0&0&0&0&1&1&2&1&0&2&2&0&1&1&2&1&0&2&2&0&1&1&2&1\\
\hline
0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&1&0\\
0&0&1&2&2&1&2&2&1&0&0&1&2&2&1&2&2&1&0&0&1&2&2&1&2&2&1\\
1&1&1&1&2&1&2&1&1&1&1&1&1&2&1&2&1&1&1&1&1&1&2&1&2&1&1\\
1&1&2&0&0&2&0&0&2&1&1&2&0&0&2&0&0&2&1&1&2&0&0&2&0&0&2\\
2&2&1&2&0&1&0&2&1&2&2&1&2&0&1&0&2&1&2&2&1&2&0&1&0&2&1\\
1&1&0&2&1&0&1&2&0&1&1&0&2&1&0&1&2&0&1&1&0&2&1&0&1&2&0\\
0&0&2&0&1&2&1&0&2&0&0&2&0&1&2&1&0&2&0&0&2&0&1&2&1&0&2\\
2&2&2&0&2&2&2&0&2&2&2&2&0&2&2&2&0&2&2&2&2&0&2&2&2&0&2\\
2&2&0&1&1&0&1&1&0&2&2&0&1&1&0&1&1&0&2&2&0&1&1&0&1&1&0
\end{array}
$$

Now to obtain a new perfect factor we integrate to obtain a new array. Row 0 of the new array is the starter and subsequent rows are obtained by adding. That is, row 1 is the starter plus row 0 of the intermediate array above. Row 2 is the starter plus rows 0 and 1, ... row i is the starter plus rows 1 through $i - 1$ of the intermediate array. Since all column sums are zero mod 3, the period of the columns stays the same.

$$
\begin{array}{ccccccccccccccccccccccccccc}
0&2&2&0&0&0&0&1&1&2&1&0&2&2&0&1&1&2&1&0&2&2&0&1&1&2&1\\
0&2&2&1&0&0&0&2&1&2&1&0&0&2&0&1&2&2&1&0&2&0&0&1&1&0&1\\
0&2&0&0&2&1&2&1&2&2&1&1&2&1&1&0&1&0&1&0&0&2&2&2&0&2&2\\
1&0&1&1&1&2&1&2&0&0&2&2&0&0&2&2&2&1&2&1&1&0&1&0&2&0&0\\
2&1&0&1&1&1&1&2&2&1&0&1&0&0&1&2&2&0&0&2&0&0&1&2&2&0&2\\
1&0&1&0&1&2&1&1&0&0&2&2&2&0&2&2&1&1&2&1&1&2&1&0&2&2&0\\
2&1&1&2&2&2&2&0&0&1&0&2&1&1&2&0&0&1&0&2&1&1&2&0&0&1&0\\
2&1&0&2&0&1&0&0&2&1&0&1&1&2&1&1&0&0&0&2&0&1&0&2&1&1&2\\
1&0&2&2&2&0&2&0&1&0&2&0&1&1&0&0&0&2&2&1&2&1&2&1&0&1&1
\end{array}
$$

Doing the same thing with the other two possible starters produces three 3-ary 9×27 arrays in which we claim that every 3-ary 3×2 subarray appears exactly once. We call this a $((9,27);(3,2);3)^2_3$-PF (perfect factor).

For example, try to find the subarray $\begin{matrix}0&0\\1&2\\2&0\end{matrix}$. Look at the differences (mod 3) between row 0 and row 1 and between row 1 and row 2. These differences give $\begin{matrix}1&2\\1&1\end{matrix}$ which occurs in position $(1,2)$ in the $((9,9);(2,2))_3$-dBT of Example 1. The sum of the entries on these two columns above $\begin{matrix}1&2\\1&1\end{matrix}$ is $0\ 1$. In the addition we need to 'arrive' at position $(1,2)$ (mod $(9,9)$) with a sum of $0\ 0$, the first row of the array we are looking for. Thus the starter plus $0\ 1$ must be $0\ 0$ (the first row of our particular subarray). So we need to find $0\ 2$ in the starter in a position 2 modulo 9. This occurs with the second starter in position 11. Thus we find $\begin{matrix}0&0\\1&2\\2&0\end{matrix}$ in row 1 column 11 of the new array. Similarly, since every length 2 substring appears in the set of 3 starters in every position modulo 9, we can find every 3×2 subarray. In this example we have integrated along columns, (the first coordinate dimension), when we describe integration in general we integrate along dimension d, so one should take the transpose of our examples to be consistent with that notation.

Example 4: In Example 2, the column sums are zero mod 3. Here we give an example of integration when the column sums are a nonzero constant. Consider the two dimensional array

$$\begin{matrix}0&3&2&2\\1&2&3&3\\2&1&1&0\\3&0&0&1\end{matrix}$$

Viewing this 4×4 array toroidally, every 1×2 4-ary subarray appears exactly once. So this is a $((4,4);(1,2))_4$-dBT (de Bruijn torus). Note that the column sums are 2 mod 4. Consider also the collection of strings

0 0 0 0	1 1 1 1	0 1 0 1	1 0 1 0	0 2 0 2	0 3 0 3	1 2 1 2	1 3 1 3
2 2 2 2	3 3 3 3	2 3 2 3	3 2 3 2	2 0 2 0	2 1 2 1	3 0 3 0	3 1 3 1

This is a set of 16 period 4 strings in which each 4-ary length 2 substring appears exactly once in each position modulo 4. We call this a $(4;2;16)_4[4]$-PMF (perfect multifactor). However there is an additional property, doing arithmetic mod 4 the two strings in each column differ by $2\ 2\ 2\ 2$. Thinking of the entries from Z_4 and $\{0,2\}$ as a subgroup of Z_4 we call this an equivalence class perfect multifactor modulo $\{0,2\}$ an denote it $(4;2;16)_{Z_4|\{0,2\}}[4]$-EPMF. Picking one string from each column we get what we call a set of representatives modulo $\{0,2\}$. Now when we integrate as in example 2, use only one representative from each column in the list. The column lengths double (since in order to get column sums 0, we need two copies of the 4×4 arrays placed vertically). Illustrating, along the lines of example 2 with $0\ 1\ 0\ 1$ as a starter we get the following, with the intermediate array on the left and the integrated array on the right

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ \hline 0 & 3 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 3 & 3 & 2 & 2 \\ 2 & 3 & 2 & 3 \\ 2 & 2 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

Note that rows i and $i+4$ (modulo 8) differ by the constant string of 2's. For example adding 2 (arithmetic modulo 4) to each of the entries of 1 2 1 2 of row 2 we get 3 0 3 0 , row 6.

Doing this with one of the strings from each column of the list of the equivalence class perfect multifactor we obtain a set of eight 8×4 arrays in which each 4-ary 2×2 subarray appears exactly once, a $((8,4);(2,2);8)_4^2$-PF.

For example, try to find the subarray $\begin{smallmatrix} 0 & 1 \\ 3 & 0 \end{smallmatrix}$. The difference (mod 4) between the rows is 3 3 which occurs in position $(1,2)$ of the de Bruijn torus. The sum of the entries above these columns in the de Bruijn torus is 2 2 . In this case we need to 'arrive' at position $(1,2)$ with sum 0 1 , the first row of the array we are looking for. Thus the starter plus 2 2 must be 0 1 . So we need to find 2 3 in a position 2 modulo 4 of a starter. This occurs in the third column of the list. However, we choose the other string 0 1 0 1 as our starter so instead of finding our array in row 1 we find it in row 5. This is because the first 4 rows of integration 'change' the 0 1 0 1 to 2 3 2 3 with 2 3 in position 2 mod 4 as needed.

Example 5: In Example 2, what we did was copy a two dimensional array several times, and used a starter such that every substring appeared in every position modulo the number of columns. Working in three dimensions, imagine the array we wish to integrate as a box. We arrange copies of the box in some rectangular pattern and overlay a two dimensional starter. The starter must have the property that every 2 dimensional subarray appears exactly once in each position modulo the size of the 'tops' of the boxes. That is, we need a two dimensional perfect multifactor. In general, we need a $(d-1)$ dimensional perfect multifactor to integrate a d dimensional perfect factor.

Example 6: We now illustrate building a 2 dimensional perfect multifactor. Begin with a $(4;2,2)_2^1[2]$-PMF (perfect multifactor) 0011 1001 , a set of 2 binary 1 dimensional strings with period 4 in which each substring of length 2 appears exactly once in each position modulo 2. We will concatenate these to form the columns of a two dimensional perfect multifactor. In the previous example we specified column shifts. Here we must specify shifts as well as a selection of which column to use. The shifts must be multiples of 2 because of the modulus, so our strings times 2 give the shifts.

Consider the following

$$\begin{array}{r|l} \text{times } 2 = \text{shifts} & 0000\ 0000\ 1111\ 1111 \\ \text{column selection} & 0011\ 1001\ 0011\ 1001 \end{array}$$

In this set of four pairs of 4-tuples, each possible shift (0 or 1) appears once with each possible pair from 00, 01, 10, 11 (which specifies the column selection) in each position modulo 2. We will call this a $(4;(1,2),4)_{(2,2)}[2]$-PMFP (perfect multifactor pair). Using this, with (the transpose of) 0011 called column #0 and (the transpose of) 1001 called column #1 we get four 4×4 arrays

$$\begin{array}{cccc} 0\,0\,1\,1 & 1\,0\,0\,1 & 0\,1\,1\,0 & 1\,1\,0\,0 \\ 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,1\,0\,1 & 0\,1\,0\,1 \\ 1\,1\,0\,0 & 0\,1\,1\,0 & 1\,0\,0\,1 & 0\,0\,1\,1 \\ 1\,1\,1\,1 & 1\,1\,1\,1 & 1\,0\,1\,0 & 1\,0\,1\,0 \end{array}$$

In the fourth array, from the fourth term $\begin{smallmatrix}1111\\1001\end{smallmatrix}$ of the PMFP, we start column 0 equal to column #1, then column 2 is column #0 shifted by $2 = 2 \cdot 1$. Column 3 is column # 0 shifted by $2 = 2 \cdot 1$ from the previous column (a total shift of 4, which is equivalent to a shift of 0 since the columns have height 4). Column 4 is column #1 shifted by $2 = 2 \cdot 1$ from the previous column. The last column is again shifted by $2 = 2 \cdot 1$ for a total shift of 0, modulo 4, which is what is needed so that we 'return' to the first column and the period of the rows remains 4.

We claim that every 2×2 subarray appears exactly once in each position modulo $(2,2)$ in one of the 4 arrays. So this is a $((4,4);(2,2);4)^2_2[(2,2)]$-PMF (perfect multifactor).

For example, to find $\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$ in position $(1,0)$ modulo $(2,2)$ first observe that the first column $\begin{smallmatrix}0\\1\end{smallmatrix}$ of our subarray appears modulo 1 in position 1 in column #0 and the second column $\begin{smallmatrix}1\\1\end{smallmatrix}$ appears modulo 1 in position 3 in column #1. The positions differ by 2. So, we must find the column pair $0,1$ along with the shift 1 (since we multiply shifts by 2) in position 0 modulo 2 in our perfect multifactor pair. This is in position 4 in the fourth set. So we find $\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$ modulo $(1,0)$ in column 2 of the fourth array. This is in position $(1,2)$ of the array.

3 Basics

In this section we give more formal definitions as well as stating necessary conditions for existence of perfect multifactors.

We will denote (non-periodic) vectors as $\vec{V} = (v_1, v_2, \ldots, v_d)$ and write $\langle \vec{V} \rangle = \Pi_{i=1}^d v_d$. Deleting the last coordinate from $\vec{V}$ will result in $\vec{V}^- = (v_1, v_2, \ldots, v_{d-1})$. We will also write $\vec{V}^+$ for $(d+1)$-dimensional vectors that agree with $\vec{V}$ on the first d coordinates, with v_{d+1} specified in each particular case. For two vectors $\vec{I} = (i_1, i_2, \ldots, i_d)$ and $\vec{J} = (j_1, j_2, \ldots, j_d)$ coordinatewise multiplication will be $\vec{I} \cdot \vec{J} = (i_1 j_1, i_2 j_2, \ldots, i_d j_d)$ and addition is ordinary vector addition $\vec{I} + \vec{J} = (i_1 + j_1, i_2 + j_2, \ldots, i_d + j_d)$.

For an array A we will denote the entry in position $\vec{I} = (i_1, i_2, \ldots, i_d)$ by $[A]_{\vec{I}}$. A periodic array A with period $\vec{R} = (r_1, r_2, \ldots, r_d)$ is an infinite array such that for all $\vec{J}$, $[A]_{\vec{J}} = [A]_{\vec{J}+\vec{R}}$. For ease of notation we will consider only indices with non-negative integral values. A fundamental block of A is an array consisting of r_i consecutive rows in the i^{th} dimension for $i = 1, 2, \ldots, d$. Repeating such a block produces A. We will sometimes refer to a fundamental block of A as A

when there is no chance of confusion. A fundamental block of a one dimensional periodic array could also be viewed as a vector.

If a matrix B of size $\vec{S}$ appears in A in positions $\vec{I}$ through $\vec{I}+\vec{S}$ we say that B appears in A at position $\vec{I}$. We say that B appears in location $\vec{J} = (j_1, j_2, \ldots, j_d)$ modulo $\vec{N} = (n_1, n_2, \ldots, n_d)$ if B appears in position $\vec{I} = (i_1, i_2, \ldots, i_d)$ and $i_x \equiv j_x \pmod{n_x}$ for $x = 1, 2, \ldots, d$. Usually when we say that B appears in position $\vec{I} = (i_1, i_2, \ldots, i_d)$ in a period $\vec{R}$ array, we will pick those i_x with $0 \leq i_x < r_i$. When we say that a subarray B of A appears exactly once, we mean exactly once in any fundamental block. When looking only at a fundamental block, addition on the subscripts in the i^{th} dimension is performed modulo r_i.

The projection of a d-dimensional periodic array A *onto* the z^{th} hyperplane in dimension d is the $(d-1)$ dimensional array consisting of entries $[A]_{\vec{I}}$ for which $i_d = z$. A projection of A *along* $\vec{J} = (j_1, j_2, \ldots, j_{d-1})$ is a one dimensional array consisting of entries $[A]_{\vec{I}}$ for $\vec{I}^- = \vec{J}$. We will refer to such projections as projections along direction d.

An array will be called K-ary if the entries are from an alphabet (set) K. If we are only concerned about the size of K and not its structure we will write k-ary where $k = |K|$. Sometimes we need additional additive structure on the alphabet. When we refer to an alphabet as a group we will assume the group is of the form $Z_{a_1} \times Z_{a_2} \times \cdots \times Z_{a_n}$ for some integers n and $a_1, a_2, \ldots, a_n$. We will also sometimes view an element of $Z_{a_1} \times Z_{a_2} \times \cdots \times Z_{a_n}$ as a length n vector with entries from Z in the obvious manner. More formally, if the term from Z_{a_i} is the congruence class $[x]$ then the i^{th} component of the vector viewed in Z is the unique integer in $\{0, 1, \ldots, a_i - 1\}$ congruent to x modulo a_i.

We will write $\gcd(a,b)$ for greatest common divisor and $\operatorname{lcm}(a,b)$ for least common multiple.

Definition 1 *A* $(\vec{R}; \vec{U}; \tau)^d_K[\vec{N}]$ Perfect Multifactor *(PMF) is a collection of τ d-dimensional periodic arrays with period $\vec{R} = (r_1, r_2, \ldots, r_d)$, with entries from an alphabet K and such that every K-ary size $\vec{U} = (u_1, u_2, \ldots, u_d)$ subarray appears exactly once in each location modulo $\vec{N} = (n_1, n_2, \ldots, n_d)$. We assume that r_i is a multiple of n_i for $i = 1, 2, \ldots, d$. Sometimes we will only be concerned about the size k of K and not its structure, in which case we will replace K with k in the notation.*

Usually PMF's are defined referring only to the size $|K|$ and not the structure of the alphabet K. We have included the structure of K in our definition because we will need additive properties in K for our constructions.

The following lemma which relates the parameters of PMFs is easily verified by equating the number of distinct positions in a fundamental block and the number of appearances of subarrays, recalling that each particular subarray of size $\vec{U}$ appears exactly once for each location modulo $\vec{N}$.

Lemma 1 *For a* $(\vec{R}; \vec{U}; \tau)^d_K[\vec{N}]$ *PMF (perfect multifactor) we have*

$$\langle \vec{R} \rangle \tau = |K|^{\langle \vec{U} \rangle} \langle \vec{N} \rangle. \tag{1}$$

Additionally, if A is a set of τ period $\vec{R}$, K-ary arrays in which each K-ary size $\vec{U}$ subarray appears at least once (i.e., in some array) in each location modulo $\vec{N}$ and if (1) holds, then A is a $(\vec{R};\vec{U};\tau)^d_K[\vec{N}]$ PMF (perfect multifactor).

Definition 2 *A $(\vec{R};\vec{U};\tau)^d_K$* Perfect Factor *(PF) is a perfect multifactor in which $\vec{N}=(1,1,\ldots,1)$. When $\tau=1$ (there is only one array) we have what is called a de Bruijn cycle in dimension 1 and a de Bruijn torus in higher dimensions. These are also called perfect maps. In this case we will write $(\vec{R};\vec{U})^d_K$-dBT. (This last notation appears only in the examples.)*

Definition 3 *Let K be a group and H a subgroup of K. An $(\vec{R};\vec{U};\tau)^d_{K|H}[\vec{N}]$* Equivalence Class Perfect Multifactor *modulo H (EPMF) is a K-ary perfect multifactor with the additional condition that the τ arrays can be partitioned into a collection of size $|H|$ parts with the arrays in part z labeled $A(z,1),A(z,2),\ldots,A(z,|H|)$ such that for all i,j there is a $c\in H$ with $A(z,j)-A(z,i)$ equal to the constant array having each entry c. A set of representatives modulo H is obtained by selecting from each part one of the arrays.*

Note that when $H=\{0\}$, the trivial group then an EPMF is just a PMF.

Definition 4 *A $(Q;(u,v),\tau)_{K,L}[N]$* Perfect Multifactor Pair *(PMFP) is a collection of τ period Q sequences consisting of ordered pairs from an alphabet $K\times L$ such that every pairing of a K-ary size u string in the first coordinate with an L-ary size v string in the second coordinate occurs in each location modulo N. We will denote a perfect multifactor pair by $(A:B)$ where A indicates the collection of sequences for the first coordinate and B the collection of sequences for the second coordinate.*

The next lemma follows in the same manner as Lemma 1.

Lemma 2 *For a $(Q;(u,v),\tau)_{K,L}[N]$ PMFP (perfect multifactor pair) we have*

$$Q\tau=|K|^u|L|^vN. \tag{2}$$

Additionally, if A is a set of τ period Q sequences of pairs from an alphabet $K\times L$ in which each K-ary length u string in the first coordinate is paired with each L-ary length v string in the second coordinate at least once (i.e., in some pair) in each location modulo N and if (2) holds, then A is a $(Q;(u,v),\tau)_{K,L}[N]$ PMFP (perfect multifactor pair).

Definition 5 *The shift operator $E^{\vec{S}}$ applied to an array A shifts the location indices so that the entry in position $\vec{S}$ appears in position $(0,0,0,\ldots)$. So, in general, $[E^{\vec{S}}(A)]_{\vec{I}}=[A]_{\vec{I}+\vec{S}}$.*

The following Lemma and conjecture have been noted numerous times and in the general form here in [8]. Part (i) of the lemma follows from Lemma 1. Part (ii) follows by considering the all 0 subarray of size $\vec{U}$ and noting that it appears exactly once in each location modulo $\vec{N}$.

Lemma 3 *Suppose there exists a* $(\vec{R};\vec{U};\tau)^d_K[\vec{N}]$*-PMF (Perfect multifactor). Then*

$$(i)\ \langle\vec{R}\rangle\tau = |K|^{\langle\vec{U}\rangle}\langle\vec{N}\rangle \text{ and}$$

$$(ii) \text{ For each } i \text{ either (a) } n_i = 1 \text{ and } r_i \geq u_i \text{ or (b) } n_i > 1 \text{ and } r_i > u_i.$$

It is conjectured that except possibly for some very 'small' values of the r_i ($r_i = u_i + 1$ for example) the necessary conditions implied by the lemma are also sufficient. For many cases, particularly in one and two dimensions, sufficiency has been shown. See for example [7], [8], [11], [14], [17], [18]. However, for two and higher dimensional multifactors and three and higher dimensions in all cases the results have been fairly restricted. As previously discussed, the new results for integration in higher dimensions in Section 4 should be a useful tool in covering more of these cases.

4 Constructions

In this section we describe two basic construction methods, concatenation and integration of perfect multifactors. We then give our main results for situations where these constructions produce new perfect multifactors. The proofs follow the same patterns that have been developed in the literature previously. Indeed they may appear shorter because we separate out the key tools of constructing perfect multifactors. Once we do the work of getting the appropriate terminology and statements, the proofs are straightforward. We hope that this will aid in avoiding redundancy in future proofs and focus the development of constructions for perfect multifactors.

4.1 Concatenation

We begin with concatenation. We briefly outline various steps towards describing this in the broadest setting. Two dimensional concatenation of binary de Bruijn cycles appears in [10] and of perfect factors in [2]. Two dimensional concatenation of perfect factors over general alphabets appears in [17]. Higher dimensional concatenation of de Bruijn cycles over general alphabets appears in [1] and of perfect factors in [6]. Two dimensional concatenation of perfect multifactors appears in [18]. A special case of creating multifactors when the shift vector does not have zero sum appears in [1], however most of what we do in this case is new. Here we also include higher dimensional concatenation of perfect multifactors. In every case, the key is 'lining' up selection of factors with shift patterns, and has been done previously by specific construction in the proof. By introducing perfect multifactor pairs, we separate out this key part of the proof, simplifying the proof for concatenation. Of course then more needs to be said about perfect multifactor pairs and we will do this below.

Construction 1 (Concatenation) Let $A = A(1), A(2), \ldots, A(\tau)$ be a collection of d-dimensional period $\vec{R}$ arrays. Let $B = B(1), B(2), \ldots, B(\rho)$ and

$C = C(1), C(2), \ldots, C(\rho)$ be collections of ρ (one-dimensional) strings, each string with period Q. The entries of the $C(i)$ are from the alphabet $\{1, 2, \ldots, \tau\}$. For some $\vec{N}$, the entries of the $B(i)$ are from the group $H = Z_{r_1/n_1} \times Z_{r_2/n_2} \times \cdots Z_{r_d/n_d}$ and satisfy the following. There exists a $c \in H$ such that for $j = 1, 2, \ldots, \rho$ we have $\displaystyle\sum_{h=1}^{Q} [B(j)]_h = c$. That is, the sum of entries in a fundamental block is c. We call the pair $(B : C)$ the *indexer*.

Then *concatenating* A using indexer $(B : C)$ produces a new collection of $(d+1)$-dimensional arrays $D(1), D(2), \ldots, D(\rho)$ each with period $\vec{R}^+$ where the first d coordinates of $\vec{R}^+$ are the same as $\vec{R}$ and r^+_{d+1} equals Q times the order of c in H.

To describe the entries, define $S(j, z) \in H$ by $S(j, z) = \sum_{h=0}^{z-1} [B(j)]_h$. The sum is 0 if $z = 0$. We can view $S(j, z)$ as a length d vector with entries in Z and write this as $\vec{S}(j, z)$. Then $\vec{N} \cdot \vec{S}(j, z) = (n_1 s_1, n_2 s_2, \ldots, n_d s_d)$. For $j = 1, 2, \ldots, \rho$ and for $\vec{I} = (i_1, i_2 \ldots, i_{d+1})$ with $i_{d+1} = z$, we have

$$[D(j)]_{\vec{I}} = [A([C(j)]_z)]_{\vec{I}^- + \vec{N} \cdot \vec{S}(j,z)} = \left[E^{\vec{I}^- + \vec{N} \cdot \vec{S}(j,z)} A([C(j)]_z)\right]_{\vec{I}}.$$

In other words, the projections of $D(j)$ onto hyperplanes in dimension $d+1$ are shifted factors from A. The selection of which factor is determined by C and the shifts are multiples of $\vec{N}$ determined by B.

To check that concatenation is well defined we only need to check that the periods are correct. For the first d dimensions this follows immediately from the observations about projections above. For dimension $d + 1$ this follows by observing that the projection onto the Q hyperplane in dimension $(d + 1)$ is shifted c 'times' $\vec{N}$ (coordinatewise multiplication) relative to the projection onto the 0 hyperplane in dimension $(d+1)$. If η is the order of c, then $\eta c = 0$ and the projection onto the $Q\eta$ hyperplane is shifted by 0 relative to the projection onto the 0 hyperplane. That is, they are shifted the same amount. Also since the $B(i)$ have period Q, the projections onto the 0 and the $Q\eta$ hyperplanes are the same factor from A.

For the next theorem, the case $c = 0$ is the one that has been covered previously. We also include $c \neq 0$ for completeness. Even though the possibilities for $c \neq 0$ are fairly restricted we believe there may be some use in constructing exceptional perfect multifactors.

Theorem 1 *Let A be a $(\vec{R}; \vec{V}; \tau)^d_G[\vec{N}]$ PMF (perfect multifactor). Let $H = Z_{r_1/n_1} \times Z_{r_2/n_2} \times \cdots Z_{r_d/n_d}$ and let $H' = \{1, 2, \ldots, \tau\}$. Let $(B : C)$ be a $(Q; (U-1, U); \rho)_{H,H'}[M]$ PMFP (perfect multifactor pair) with the following property. There exists $c \in H$ such that each string $B(j)$ in B satisfies $\displaystyle\sum_{h=1}^{Q} [B(j)]_h = c$. That is, the entries in each fundamental block sum to c.*

Then,

- *If* $c = 0 \in H$*, concatenation using* $(B : C)$ *as indexer yields a* $(\vec{R}^+; \vec{V}^+; \rho)_G^{d+1}[\vec{N}^+]$ *PMF (perfect multifactor) where the first* d *coordinates of* $\vec{N}^+$*,* $\vec{R}^+$ *and* $\vec{V}^+$ *are the same as* $\vec{N}$*,* $\vec{R}$ *and* $\vec{V}$ *and* $n^+_{d+1} = M$*,* $r^+_{d+1} = Q$ *and* $v^+_{d+1} = U$*.*
- *If* $c \neq 0 \in H$ *and additionally we have the following: If* c *is viewed as a vector* $\vec{C} = (c_1, c_2, \ldots, c_d)$ *with entries from* Z *and for* $i = 1, 2, \ldots, d$ *we have* $\eta_i = \dfrac{r_i/n_i}{\gcd(r_i/n_i, c_i)}$ *(i.e., the order of* c_i *in* Z_{r_i/n_i} *is* η_i*), then* $\gcd(\eta_i, c_i) = 1$*. Also, for* $i \neq j$*,* $\gcd(\eta_i, \eta_j) = 1$*. Then concatenation using* $(B : C)$ *as indexer yields a* $(\vec{R}^+; \vec{V}^+; \rho)_G^{d+1}[\vec{N}^*]$ *PMF (perfect multifactor) where the first* d *coordinates of* $\vec{R}^+$ *and* $\vec{V}^+$ *are the same as* $\vec{R}$ *and* $\vec{V}$ *with* $r^+_{d+1} = Q\Pi_{i=1}^d \eta_i$ *and* $v^+_{d+1} = U$*. Also* $\vec{N}^*$ *is given by* $n^*_j = n_i\eta_i$ *for* $j = 1, 2, \ldots, d$ *and* $n^*_{d+1} = M$*.*

Proof: Observe that when $c = 0$ each $\eta_i = 1$. Thus $c = 0$ is included also in the $c \neq 0$ case and we need only consider $c \neq 0$. Let D denote the PMF formed by concatenation.

The periodicity and the number of factors ρ for D follow from the discussion that concatenation is well defined and from the definition of concatenation. We need only observe that for $c \neq 0$ as in the statement of the theorem, the order of c in H is $\Pi_{i=1}^d \eta_i$.

By Lemma 1 we need only to check that equation (1) holds and that each subarray of size $\vec{V}^+$ appears at least once in some array $D(j)$ of D in each location modulo $\vec{N}^*$.

From Lemma 1 applied to A and Lemma 2 applied to $(B : C)$ we have $\langle\vec{R}\rangle\tau = |G|^{\langle\vec{V}\rangle}\langle\vec{N}\rangle$ and $Q\rho = |H|^{U-1}|H'|^U M = \left(\dfrac{\langle\vec{R}\rangle}{\langle\vec{N}\rangle}\right)^{U-1}\tau^U M$. Then with $\langle\vec{R}^+\rangle = \langle\vec{R}\rangle r_{d+1} = \langle\vec{R}\rangle Q\Pi_{i=1}^d\eta_i$ and $\langle\vec{V}^+\rangle = \langle\vec{V}\rangle v_{d+1} = \langle\vec{V}\rangle U$ and $\langle\vec{N}^*\rangle = \Pi_{i=1}^{d+1} n^*_i = n^*_{d+1}\Pi_{i=1}^d n_i\eta_i = M\langle\vec{N}\rangle\Pi_{i=1}^d\eta_i$ we get

$$\begin{aligned}
\langle\vec{R}^+\rangle\rho &= \langle\vec{R}\rangle\left(\Pi_{i=1}^d\eta_i\right)Q\rho \\
&= \langle\vec{R}\rangle\left(\Pi_{i=1}^d\eta_i\right)\left(\frac{\langle\vec{R}\rangle}{\langle\vec{N}\rangle}\right)^{U-1}\tau^U M \\
&= \left(|G|^{\langle\vec{V}\rangle}\langle\vec{N}\rangle\right)^U \frac{M\Pi_{i=1}^d\eta_i}{\langle\vec{N}\rangle^{U-1}} \\
&= |G|^{\langle\vec{V}^+\rangle}\langle\vec{N}^*\rangle.
\end{aligned}$$

So D satisfies equation (1).

Now, for an arbitrary G-ary size $\vec{V}^+$ array D' and an arbitrary location $\vec{L}^*$ we must find D' in some position $\vec{I} \equiv \vec{L}^*$ modulo $\vec{N}^*$ in some factor of D.

Let D'_h be the projection of D' onto the h hyperplane in dimension $(d+1)$ for $h = 0, 1, \ldots, (U-1)$. If $\vec{L}^* = (l^*_1, l^*_2, \ldots, l^*_{d+1})$ let $l'_i \equiv l^*_i \pmod{n_i}$ for $i =$

$1, 2, \ldots, d$ and let $\vec{L}' = (l'_1, l'_2, \ldots, l'_d)$. Then D'_h appears in position $\vec{J}'(h) \equiv \vec{L}'$ (mod $\vec{N}$) in some factor $A(f(h))$ of A. Since the $\vec{J}'(h) \equiv \vec{L}'$ (mod $\vec{N}$), for $h = 0, 1, \ldots, (U-2)$ we have $\vec{J}'(h+1) - \vec{J}'(h) = (v_1, v_2, \ldots, v_d)_h \cdot (n_1, n_2, \ldots, n_d) = \vec{V}_h \cdot \vec{N}$ where the v_i can be viewed as elements of Z_{r_i/n_i} and the $\vec{V}_h$ as elements V_h of $H = Z_{r_1/n_1} \times Z_{r_2/n_2} \times \cdots Z_{r_d/n_d}$. Hence $(V_1, V_2, \ldots, V_{U-2})$ is a length $(U-1)$ string in H. Then $(V_1, V_2, \ldots, V_{U-2})$ appears together with $(f(0), f(1), \ldots, f(U-1))$ in position $l'_{d+1} \equiv l^*_{d+1}$ (mod M) in some factor $(B : C)(j)$ of $(B : C)$.

Then, D' appears in position $\vec{I}'$ in $D(j)$ where $i'_{d+1} = l'_{d+1}$ and for $x = 1, 2, \ldots, d$, $i'_x \equiv l'_x$ (mod n_i). In fact, $\vec{I}'$ is such that $\vec{I}^- + \vec{N} \cdot \vec{S}(j, l'_{d+1}) = \vec{J}'(0)$. Now, since gcd $(\eta_i, c_i) = 1$, there exists $z_x \in \{0, 1, \ldots, \eta_x - 1\}$ with $i'_x + z_x n_x c_x \equiv l^*_x$ (mod n^*_x). (Recall $n^*_x = n_x \eta_x$.) Since, for $x \neq y$, gcd$(\eta_x, \eta_y) = 1$, there exists $m \in \{0, 1, \ldots, \Pi_{i=1}^d \eta_i\}$ with $mc = (z_1, z_2, \ldots, z_d)$ (in H). Then D' appears in D in position $\vec{I}$ with $i_x = i'_x + z_x n_x c_x$ for $x = 1, 2, \ldots, d$ and $i_{d+1} = l'_{d+1} + mQ$. For $x = 1, 2, \ldots, d$ we already have $i_x = i'_x + z_x n_x c_x \equiv l^*_x$ (mod n^*_x) from above. Also from $(B : C)$ we have Q a multiple of $M = n^*_{d+1}$. Along with $l'_{d+1} \equiv l^*_{d+1}$ (mod M) we get $i_{d+1} \equiv l^*_{d+1}$ (mod n^*_{d+1}). Hence, $\vec{I}' \equiv \vec{L}^*$ (mod $\vec{N}^*$) as needed. □

4.2 Integration

As with concatenation we will briefly outline the various steps toward the application of integration in the broad setting given here. One dimensional integration in the context of binary perfect factors was first discussed by [9] and extended to prime power alphabets with a discussion of repeated application in [2] and [16]. Two dimensional integration of de Bruijn tori in the binary case appears in [3]. Two dimensional integration of de Bruijn tori over general alphabets appears in [5] and [17]. In [17] there is further discussion of complexity in the new tori to allow repeated application of integration. The binary two dimensional case with constant (non-zero) sums is discussed in [3]. Although integration of factors and multifactors in two dimensions has not been discussed it is the same as for perfect maps. Here we include integration in dimensions three and higher. Previous proofs have included construction of specific perfect multifactors for starters. Our proof is essentially the same as previous proofs, but by making explicit the role of perfect multifactors as starters for integration the proof appears simpler. The construction of perfect multifactors is covered by our concatenation result and inductively by the integration result. Additionally, our approach to multifactors as starters is what allows the extension to higher dimensions. We describe integration along direction d to simplify notation. To integrate along other directions simply 'transpose' the dimensions.

Construction 2 (Integration) Let $A = A(1), A(2), \ldots, A(\rho)$ be a collection of G-ary d-dimensional period $\vec{Q}$ arrays with the sum of entries in each (one dimensional) projection along direction d equal to a constant $c \in G$. Let $B = B(1), B(2), \ldots, B(\tau)$ be a collection of $(d-1)$-dimensional period $\vec{R}$ arrays with

r_i a multiple of q_i for $i = 1, 2, \ldots, (d-1)$. We call B the *starter*.

Then *integrating* A with starter B produces a new collection $D(i, j)$ (for $i = 1, 2, \ldots, \rho$ and $j = 1, 2, \ldots, \tau$) of d-dimensional arrays with period $\vec{R}^+$ where the first $(d-1)$ coordinates of $\vec{R}^+$ are the same as $\vec{R}$ and $r_d^+ = \eta q_d$ where η is the order of c in G.

To describe the entries, let $\vec{e}(d)$ be the d-dimensional unit vector in direction d, i.e., $\vec{e}(d) = (0, 0, \ldots, 0, 1)$. For $\vec{I} = (i_1, i_2 \ldots, i_d)$ with $i_d = z$, we have

$$[D(i,j)]_{\vec{I}} = [B(j)]_{\vec{I}^-} + \sum_{h=1}^{z} [A(i)]_{\vec{I}-h\vec{e}(d)}.$$

The sum term is zero when $z = 0$.

In other words, looking at a projection along direction d, the entry in position zero is from the starter and the i^{th} entry is the sum of the starter entry plus entries in positions zero to $i-1$ from the corresponding projections along d in A. So the i^{th} entry of a projection $(i > 0)$ in D is the $(i-1)^{st}$ entry of D plus the $(i-1)^{st}$ entry of the corresponding projection of A.

One dimensional integration of a sequence $(a_0, a_1, a_2, \ldots)$ with starter s produces the sequence $(s, s+a_0, s+a_0+a_1, s+a_0+a_1+a_2, \ldots,)$. For d dimensional integration we integrate each one dimensional projection along direction d with starter from the appropriate position in the $(d-1)$ dimensional starter.

To check that integration is well defined we only need to check that the periods are correct. For the first $(d-1)$ dimensions this follows immediately from the periodicity of A and of the starter B. For dimension d this follows by observing that the projection along d is the starter plus sums of entries from the projection along d in A. The projection onto the 0 hyperplane in dimension d is an entry s from the starter. The projection onto the q_d hyperplane in dimension d is s plus c since we sum along all entries of A. Then the projection onto the ηq_d hyperplane is $s + \eta c = s$ and D has period (in dimension d) ηq_d.

As with concatenation, the case $c = 0$ in the following theorem is probably the most important but we include the general case for completeness. For certain situations the $c \neq 0$ case may prove valuable.

Theorem 2 *Let A be a $(\vec{Q}; \vec{U}; \rho)^d_G[\vec{N}]$ PMF (perfect multifactor) with the sum of entries in each (one dimensional) projection along direction d equal to a constant $c \in G$. Let $\vec{Q}^-$ and $\vec{U}^-$ be obtained from $\vec{Q}$ and $\vec{U}$ by deleting the d^{th} dimension.*

Then,

- *If $c = 0$, let B be a $(\vec{R}; \vec{U}^-; \tau)^{d-1}_{G|H}[\vec{Q}^-]$ EPMF (equivalence class perfect multifactor modulo H). Integrating A with starter B yields a $(\vec{R}^+; \vec{U}^*; \rho\tau)^d_{G|H}[\vec{N}]$ EPMF (equivalence class perfect multifactor modulo H) where $\vec{U}^* = \vec{U} + \vec{e}(d)$ and $r_d^+ = q_d$.*
- *If $c \neq 0$, let H' be the subgroup generated by c. Let B be a set of representatives modulo H' of a $(\vec{R}; \vec{U}^-; \tau)^{d-1}_{G|H'}[\vec{Q}^-]$ EPMF (equivalence class*

perfect multifactor modulo H'). Integrating A with starter B yields a $(\vec{R}^{+};\vec{U}^{*};\rho\tau/|H'|)^{d}_{G}[\vec{N}]$ PMF (perfect multifactor) where $\vec{U}^{*} = \vec{U} + \vec{e}(d)$ and $r^{+}_{d} = |H'|q_d$.

Proof: Observe that when $c = 0$, $|H'| = 1$. Thus, except for the equivalence class modulo H portion, the $c = 0$ case is included in the case $c \neq 0$. So for now we will consider $c \neq 0$. Let D denote the PMF formed by integration.

The periodicity and the number of factors $\rho\tau/|H'|$ for D follow from the discussion that integration is well defined and from the definition of integration.

By Lemma 1 we need only to check that equation (1) holds and that each subarray of size $\vec{U}^{*}$ appears at least once in some array $D(i,j)$ of D in each location modulo $\vec{N}$.

From Lemma 1 applied to A and B we have $\langle\vec{Q}\rangle\rho = |G|^{\langle\vec{U}\rangle}\langle\vec{N}\rangle$ and $\langle\vec{R}\rangle\tau = |G|^{\langle\vec{U}^{-}\rangle}\langle\vec{Q}^{-}\rangle$. Then, with $\langle\vec{Q}\rangle = \langle\vec{Q}^{-}\rangle q_d$ and $\langle\vec{R}^{+}\rangle = \langle\vec{R}\rangle r^{+}_{d} = \langle\vec{R}\rangle|H'|q_d$ and $\langle\vec{U}^{*}\rangle = \Pi^{d}_{i=1}u^{*}_{i} = u^{*}_{d}\Pi^{d-1}_{i=1}u^{*}_{i} = (u_d+1)\Pi^{d-1}_{i=1}u_i = \langle\vec{U}\rangle + \langle\vec{U}^{-}\rangle$ we get

$$\begin{aligned}
\langle\vec{R}^{+}\rangle\frac{\rho\tau}{|H'|} &= \langle\vec{R}\rangle\frac{\rho\tau|H'|q_d}{|H'|}\\
&= |G|^{\langle\vec{U}^{-}\rangle}\langle\vec{Q}^{-}\rangle\frac{|G|^{\langle\vec{U}\rangle}\langle\vec{N}\rangle}{\langle\vec{Q}\rangle}q_d\\
&= |G|^{\langle\vec{U}^{-}\rangle+\langle\vec{U}\rangle}\langle\vec{N}\rangle\frac{\langle\vec{Q}^{-}\rangle q_d}{\langle\vec{Q}\rangle}\\
&= |G|^{\langle\vec{U}^{*}\rangle}\langle\vec{N}\rangle.
\end{aligned}$$

So D satisfies equation (1).

Now, for an arbitrary G-ary size $\vec{U}^{*}$ array D' and an arbitrary location $\vec{L}$ we must find D' in some position $\vec{I} \equiv \vec{L}$ modulo $\vec{N}$ in some factor of D.

Let D'_h be the projection of D' onto the h hyperplane in dimension d for $h = 0,1,\ldots,u^{*}_{d}-1$. Let D'' be the size $\vec{U}$ array with projection D''_h onto the h hyperplane in dimension d equal to $D'_{h+1} - D'_h$ for $h = 0,1,\ldots,u_d-1$ (recall $u_d = u^{*}_{d}-1$). D'' appears in position $\vec{I}'' \equiv \vec{L} \pmod{\vec{N}}$ in some factor $A(j)$ of A since A is a PMF. Let $A(j)_h$ be the projection of $A(j)$ onto the h hyperplane in dimension d and let A'' be the size $\vec{U}^{-}$ array in position $\vec{I}''^{-}$ of $\sum_{h=0}^{i''_d-1} A(j)_h$.

Let B' be the size $\vec{U}^{-}$ array with $B' = D'_0 - A''$.

Now, since B is a set of representatives modulo H' for the PMF B, there exists some $c'' \in H'$ and x and B'' with $B'' - B'$ the constant array with all entries c'' and with B'' appearing in some position $\vec{I}' \equiv \vec{I}''^{-} \pmod{\vec{Q}^{-}}$ in the factor $B(x)$ selected as one of the representatives modulo H'. Since H' is generated by c and since $c'' \in H'$, we have $c'' = \gamma c$ for some $\gamma \in \{0,1,\ldots,|H'|-1\}$.

Then D' appears in position $\vec{I} \equiv \vec{L} \pmod{\vec{N}}$ in factor $D(j,x)$ of D. Here, for $h = 1,2,\ldots(d-1)$, $i_h = i'_h$ and $i_d = i''_d + \gamma q_d$. By $\vec{I}'' \equiv \vec{L} \pmod{\vec{N}}$ and

$\vec{I}' \equiv \vec{I}''$ (mod $\vec{Q}$) with q_x a multiple of n_x (since A is a PMF) for $x = 1, 2, \ldots, d$ we get $\vec{I} \equiv \vec{L}$ (mod $\vec{N}$).

With $\vec{I}$ as in the previous paragraph and using the definition of integration we get

$$\begin{aligned} [D(j,x)]_{\vec{I}} &= [B(x)]_{\vec{I}^-} + \sum_{h=1}^{i''_d+\gamma q_d} [A(j)]_{\vec{I}-h\vec{e}(d)} \\ &= [B(x)]_{\vec{I}'} + \gamma c + \sum_{h=1}^{i''_d} [A(j)]_{\vec{I}-h\vec{e}(d)} \end{aligned}$$

since the sum of entries along direction d in a fundamental block is c. Recalling the alternate view on integration in term of projections we see that the size $\vec{U}^-$ array in position $\vec{I}^-$ of the projection of $D(j,x)$ onto the i''_d hyperplane in dimension d is the sum of size $\vec{U}^-$ arrays in position $\vec{I}^-$ from $B(x)$ and from the projections of $A(j)$ onto the $i''_d - h$ hyperplanes in dimension d for $h = 1, 2, \ldots, i''_d$ plus an array with all entries $\gamma c = c''$. That is, the array is B'' plus $\sum_{h=0}^{i''_d-1} A(j)_h$ plus the array with all entries c''. But this is just D'_0. Again using the description of integration in terms of projections, we see that the projection of $D(j,x)$ onto the $i_d + 1$ hyperplane is D'_0 plus the projection of $A(j)$ onto the i''_d hyperplane. This is D'_0 plus $D'_1 - D'_0$ or just D'_1. Then, generally for $z = 1, 2, \ldots, u^*_d$, the projection of $D(j,x)$ onto the i_d+z hyperplane is $D'_0 + \sum_{h=0}^{z-1} A(j)_{i''_d+h} = D'_0 + \sum_{h=0}^{z-1} D''_h = D'_0 + \sum_{h=0}^{z-1}(D'_{h+1} - D'_h) = D'_z$. So D' appears in position $\vec{I}$ in $D(j,x)$.

When $c = 0$ we need to show that the $D(i,j)$ can be partitioned into parts of size $|H|$ with difference a constant as required for an EPMF. Note that $[D(i,j)]_{\vec{I}} - [D(i,j')]_{\vec{I}} = [B(j)]_{\vec{I}^-} - [B(j')]_{\vec{I}^-}$. So an equivalence partition on B carries over to a partition on D. □

Observe that when $c = 0$ we could use just a PMF for a starter since we could pick H to be the trivial subgroup. Observe also that when $c \neq 0$ we use only a set of representatives for a starter, so we do do not have the partitions on B and hence do not get D to be an EPMF.

Consider also repeated applications of integration. Something can be said along the lines of [17] when the alphabet is a finite field. Also, in [14] repeated application of integration is discussed when the alphabet is Z_c where c is square free. These results hold promise to carry over in certain cases and should be quite useful. However in the most general setting here involving equivalence classes, multifactors and general alphabets the results do not carry over. Observe that if the entries are from $Z_{a_1} \times Z_{a_2} \times \cdots Z_{a_l}$ with $W = \mathrm{lcm}(a_1, a_2, \ldots, a_l)$ then in any dimension in which r_i/q_i is a multiple of W, the sums will be zero and we will be able to integrate the new factor along that direction. In three and higher dimensions this should allow quite a bit of flexibility for repeated integration.

4.3 Perfect Multifactor Pairs

Here we briefly discuss perfect multifactor pairs. We give one simple construction that is essentially the one used (although without this notation) within previous proofs for integration in the literature. See [7] for another type of construction in a special case.

Lemma 4 *Let A be a $(R; u; \tau)^1_G[N]$ PMF (perfect multifactor) and let B be a $(Q; v; \rho)^1_H[R]$ PMF (perfect multifactor). Let $(A : B)$ denote the set of $\tau\rho$ sequences of ordered pairs obtained by pairing each sequence of A with each sequence of B. Then $(A : B)$ is a $(Q; (u, v); \tau\rho)_{G,H}[N]$ PMFP (perfect multifactor pair).*

Proof: Observe that a fundamental block of a sequence in $(A : B)$ consists of one fundamental block of B in the second coordinate along with Q/R copies of one fundamental block of A in the first coordinate. So the pairs have period Q. There are τ sequences in A and ρ sequences in B so the number of pairs of sequences is $\tau\rho$. By Lemma 1, we have $R\tau = N|G|^u$ and $Q\rho = R|H|^v$ from the PMFs A and B. Thus we have $Q\tau\rho = N|G|^u|H|^v$ as required for a PMFP in Lemma 2. By Lemma 2, it remains to show that each length u string μ in A is paired with each length v string ν in B in each location z modulo N. Note that N divides R since A is a PMF. We know that μ appears in position $x \equiv z \pmod N$ in some sequence $A(i)$ in A. Also, ν appears in position $y \equiv x \pmod R$ in some $B(j)$. Then the pair $(\mu : \nu)$ appears in position $y \equiv z \pmod N$ in the sequence obtained by pairing $A(i)$ and $B(j)$. □

General results for perfect multifactors needed to construct PMFPs can be found in [19] and [11]. With care in the choice of PMFs we can get PMFPs with the additional properties in terms of zero sums needed for concatenation.

5 Conclusion

We have described in a broad setting two techniques, concatenation and integration that have been used for constructing higher dimensional analogues of de Bruijn cycles (perfect maps). These methods also construct perfect factors and multifactors which are inductively used in the constructions. We have new results for using concatenation to produce perfect multifactors as well as the first description of integration in dimensions three and higher. This framework sets the stage for constructing large new families of perfect multifactors in higher dimensions, although many details remain to carry this program out completely.

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