Math 163 - Introductory Seminar
Lehigh University Spring 2008

Notes on Fibonacci numbers, binomial coefficients and mathematical induction.

These are mostly notes from a previous class and thus include some material not covered in Math 163. For completeness this extra material is left in the notes.

Observe that these notes are somewhat informal. Not all terms are defined and not all proofs are complete. The material in these notes can be found (with more detail in many cases) in many different textbooks.

Fibonacci Numbers

What are the Fibonacci numbers?

The Fibonacci numbers are defined by the simple recurrence relation

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ with } F_0 = 0, F_1 = 1.$$

This gives the sequence $F_0, F_1, F_2, \ldots = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$. Each number in the sequence is the sum of the previous two numbers. We read F_0 as 'F naught'.

These numbers show up in many areas of mathematics and in nature. For example, the numbers of seeds in the outermost rows of sunflowers tend to be Fibonacci numbers. A large sunflower will have 55 and 89 seeds in the outer two rows.

Can we easily calculate large Fibonacci numbers without first calculating all smaller values using the recursion?

Surprisingly, there is a simple and non-obvious formula for the Fibonacci numbers. It is:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

It is not immediately obvious that this should even give an integer. Since $-1 < \frac{1-\sqrt{5}}{2} < 0$ (it is approximately $-.618$) the second term approaches 0 as n gets large. Thus the first term is a good approximation of the Fibonacci numbers. In fact

$$F_n \text{ is the integer closest to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

The Golden Ratio

The number $\frac{1+\sqrt{5}}{2}$ shows up in many places and is called the Golden ratio or the Golden mean. For one example, consider a rectangle with height 1 and width x. If a vertical line is drawn in the middle so that the left side is a square and the right side is a smaller rectangle proportional to the original then x is the golden ratio.

To see this, note that for the rectangles to be proportional, the ratios of the longer sides to the smaller are equal. That is $x/1 = 1/(x-1)$. So $x(x-1) = 1 \Rightarrow x^2 - x - 1$. Using the quadratic formula x must be $(1+\sqrt{5})/2$ or $(1-\sqrt{5})/2$. Since the second is negative, x is the first term, the golden ratio. This proportion for rectangles shows up for example in the dimensions of the Parthenon and in art. There is no evidence that the ancient Greek used the golden ratio in planning the parthenon. Rather the shape seems to be aesthetically pleasing. Da Vinci did sketches of the golden ration in relation to the human body. There is debate as to whether or not he incorporated these ideas into his paintings. Dali explicitly used the golden ratio in some of his paintings.

Other examples of recurrences

Consider two other examples of simple recurrences

If $A_n = 2A_{n-1} - 2A_{n-2}$ for $n \geq 2$ with $A_0 = 2$ and $A_1 = 2$ we get the sequence $2, 2, 0, -4, -8, -6, 0, 16, 32, 32, 0, \ldots$. A number in this sequence is twice the previous number minus twice the number preceding the previous number. The general expression for A_n is even more surprising than that for Fibonacci numbers: $A_n = (1+i)^n + (1-i)^n$ where $i = \sqrt{-1}$.

A simpler example that will be useful for illustration is $B_n = B_{n-1} + 6B_{n-2}$ for $n \geq 2$ with $B_0 = 1$ and $B_1 = 8$. This recurrence gives the sequence $1, 8, 14, 62, 146, 518, \ldots$. The general formula is $B_n = 2 \cdot 3^n + (-1)(-2)^n$.

Mathematical Induction

Later we will see how to easily obtain the formulas that we have given for F_n, A_n, B_n. For now we will use them to illustrate the method of mathematical induction. We can prove these formulas correct once they are given to us even if we would not know how to discover the formulas.

First we give a proof using the idea of contradiction and that of a minimal counterexample. This is essentially the same as what we will do with induction but using slightly different language.

Proposition: If $B_n = B_{n-1} + 6B_{n-2}$ for $n \geq 2$ with $B_0 = 1$ and $B_1 = 8$ then $B_n = 2 \cdot 3^n + (-1)(-2)^n$.

Proof (using the method of minimal counterexamples): We prove that the formula is correct by contradiction. Assume that the formula is false. Then there is some smallest value of n for which it is false. Calling this value k we are assuming that the formula fails for k but holds for all smaller values. That is, we assume that $B_k \neq 2 \cdot 3^k + (-1)(-2)^k$ but that $B_{k-1} = 2 \cdot 3^{k-1} + (-1)(-2)^{k-1}$, $B_{k-2} = 2 \cdot 3^{k-2} + (-1)(-2)^{k-2}, \ldots, B_0 = 2 \cdot 3^0 + (-1)(-2)^0$. Note first that substituting $n = 0$ and $n = 1$ into the formula we get $B_0 = 2 \cdot 3^0 + (-1)(-2)^0 = 1$ and $B_1 = 2 \cdot 3^1 + (-1)(-2)^1 = 8$. So the formula works for $n = 0, 1$. Thus $k \geq 2$ and we can apply the recursion to B_k. Then using the recursion and the assumptions of the correctness of the formulas for B_{k-1} and B_{k-2} we get

$$\begin{aligned}
B_k &= B_{k-1} + 6B_{k-2} && (1)\\
&= \left[2\cdot 3^{k-1} + (-1)(-2)^{k-1}\right] + 6\left[2\cdot 3^{k-2} + (-1)(-2)^{k-2}\right] && (2)\\
&= 2\,(3+6)\,3^{k-2} + (-1)\,(-2+6)\,(-2)^{k-2} && (3)\\
&= 2\cdot 3^2\cdot 3^{k-2} + (-1)\cdot(-2)^2\cdot(-2)^{k-2} && (4)\\
&= 2\cdot 3^k + (-1)(-2)^k && (5)
\end{aligned}$$

Thus the formula holds for k, contradicting the assumption that k is the smallest number for which the formula fails. So there can be no such number: the formula holds for all $n = 0, 1, 2, \ldots$. $\square$

The key to the proof was showing that if the formula is correct for B_{k-2} and B_{k-1} then it is correct for B_k. These are the displayed computations above. Those computations work for any k. So we could use them along with the easily checked fact that the formula is correct for B_0 and B_1 to show that it is correct for B_2. Then since it is correct for B_1 and B_2 we use the computations above to see that the formula is correct for B_3. Continuing this way we see that we can build up to showing that the formula is correct for any B_n.

We display this as follows:

Formula holds for B_0 and B_1	$\Rightarrow$ (using equations (1)-(5)	$\Rightarrow$	formula holds for B_2)
Formula holds for B_1 and B_2	$\Rightarrow$ (using equations (1)-(5)	$\Rightarrow$	formula holds for B_3)
Formula holds for B_2 and B_3	$\Rightarrow$ (using equations (1)-(5)	$\Rightarrow$	formula holds for B_4)
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Formula holds for B_{k-2} and B_{k-1}	$\Rightarrow$ (using equations (1)-(5)	$\Rightarrow$	formula holds for B_k)
$\vdots$	$\vdots$	$\vdots$	$\vdots$

It is apparent that this shows that this shows that the formula works for all $n = 0, 1, 2, \ldots$.

The language used by mathematicians to describe and present this process is called mathematical induction. This provides a succinct way of stating the ideas described above. Technically it is equivalent to the minimal counterexample proof we gave above but in many situations the presentation is cleaner and shorter.

We present the same proof using the terminology of mathematical induction.

Proposition: If $B_n = B_{n-1} + 6B_{n-2}$ *for* $n \geq 2$ *with* $B_0 = 1$ *and* $B_1 = 8$ *then* $B_n = 2\cdot 3^n + (-1)(-2)^n$.

Proof (using mathematical induction): We prove that the formula is correct using mathematical induction. Since $B_0 = 2\cdot 3^0 + (-1)(-2)^0 = 1$ and $B_1 = 2\cdot 3^1 + (-1)(-2)^1 = 8$ the formula holds for $n = 0$ and $n = 1$. For $n \geq 2$, by induction

$$\begin{aligned}
B_n &= B_{n-1} + 6B_{n-2}\\
&= \left[2\cdot 3^{n-1} + (-1)(-2)^{n-1}\right] + 6\left[2\cdot 3^{n-2} + (-1)(-2)^{n-2}\right]\\
&= 2\,(3+6)\,3^{n-2} + (-1)\,(-2+6)\,(-2)^{n-2}\\
&= 2\cdot 3^2\cdot 3^{n-2} + (-1)\cdot(-2)^2\cdot(-2)^{n-2}\\
&= 2\cdot 3^n + (-1)(-2)^n
\end{aligned}$$

Hence the formula holds for all $n = 0, 1, \ldots$. $\Box$

The words 'by induction' (sometimes 'by the induction hypothesis' is used) are shorthand for the idea described above that we have already proved the statement for smaller values of n and are presenting the method to get to the next value and that this can be repeated to show the statement for all n.

Induction and the well ordering principle

Formal descriptions of the induction process can appear at first very abstract and hide the simplicity of the idea. For completeness we give a version of a formal description of mathematical induction and also that of the well ordering principle on which the minimal counterexample proof was based. The two are equivalent in that if either is assumed as a basic axiom then the other can be shown to follow from the axiom.

Well ordering principle - Every non-empty subset of integers contains a smallest element.

Principle of mathematical induction - If S_n is a statement about the positive integer n such that S_1 is true and S_k is true whenever S_{k-1} is true then S_n is true for all positive integers.

Solving linear recurrences

The example above with the Golden ratio and rectangles involved the quadratic $x^2 - x - 1 = 0$. We now explain how this same equation appears in finding a formula for the Fibonacci numbers. This method works in general for the sorts of recurrences we gave in the examples. These are linear: Terms like F_{n-1} appear by themselves and not together or in a function. So for example $F_{n-1}F_{n-2}$ or F_{n-1}^{13} or $\sin(F_{n-1})$ are not allowed in a linear recurrence. These are also homogeneous: there is no term like n^2 or 2^n not involving some F_k.

Consider what would happen if we had a solution to the Fibonacci recurrence of the form $F_n = c\alpha^n$ where $c \neq 0$ is some constant and $\alpha \neq 0$. Substituting into the recurrence we get $c\alpha^n = c\alpha^{n-1} + c\alpha^{n-2} \Rightarrow \alpha^2 = \alpha + 1$. Hence $\alpha^2 - \alpha - 1 = 0$. That is, α is a root of the quadratic $x^2 - x - 1$. Multiples and sums of functions that satisfy the recurrence will also satisfy it. So for Fibonacci numbers, both roots of $x^2 - x - 1 = 0$ satisfy the recurrence. Multiplying each by a constant and adding we get that $\lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfies the recurrence. If we can pick λ_1 and λ_2 so that we get the correct values when $n = 0$ and $n = 1$ then in fact it will be 'the' solution correct for all values. Solving $F_0 = 0 = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^0$ and $F_1 = 1 = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^1$ we get $\lambda_1 = \frac{1}{\sqrt{5}}$ and $\lambda_2 = \frac{-1}{\sqrt{5}}$ and hence the formula $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.

The Lucas numbers satisfy the same recursion as the Fibonacci numbers but have different initial conditions. They are given by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$. So the sequence is $2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots$. From the discussion above, as with the Fibonacci numbers they will be of the form $L_n = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ but with different values of λ_1 and λ_2. For the Lucas numbers $\lambda_1 = \lambda_2 = 1$ and we have $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$.

For the recurrences $A_n = 2A_{n-1} - 2A_{n-2}$ and $B_n = B_{n-1} + 6B_{n-2}$ we get that solutions will be roots of $x^2 - 2x + 2 = 0$ and $x^2 - x - 6 = 0$ respectively. The roots of the first are $1+i$ and $1-i$ and the roots of the second are 3 and -2 and these are what we see multiplied by a constant in the expressions we gave previously. Using tools from linear algebra it can be shown that if the corresponding polynomial has degree d and its roots are distinct then d initial conditions will suffice to give a formula in a manner analogous to that described above. When there are repeated roots things are only slightly more complicated. As an example $C_n = C_{n-1} + 16C_{n-2} + 20C_{n-3}$ for $n \geq 3$ with $C_0 = 1, C_1 = 10, C_2 = 54$ has the corresponding polynomial $x^3 - x^2 - 16x - 20 = (x+2)^2(x-5)$. So the roots are -2 (which is repeated) and 5. The general formula is $C_n = (-1)(-2)^n + n(-2)^n + 2 \cdot 5^n$.

Combinatorial description of Fibonacci numbers

We have already noted that Fibonacci numbers show up both in nature and in many areas of mathematics. The following five closely related objects are counted by Fibonacci numbers.

The Fibonacci numbers F_n count each of the following:
(a) the number of subsets of $\{1, 2, \ldots, n-2\}$ with no consecutive elements. (That is, subsets that do not contain i and $i+1$ for any i.)
(b) the number of binary string of length $n-2$ with no consecutive 1's. (That is, lists of 0's and 1's such no two 1's appear next to each other.)
(c) the number of lists of 1's and 2's with sum $n-1$. (The lists can have any length as long as the sum of the entries is $n-1$.)
(d) the number of lists of odd positive integers with sum n.
(e) the number of lists of integers greater than 1 with sum $n+1$.

We illustrate with $n = 6$, where $F_6 = 8$.

(a)	(b)	(c)	(d)	(e)
$\emptyset$	0000	11111	111111	7
{1}	1000	2111	3111	25
{2}	0100	1211	1311	34
{3}	0010	1121	1131	43
{4}	0001	1112	1113	52
{1,3}	1010	221	51	223
{1,4}	1001	212	33	232
{2,4}	0101	122	15	322

So column (a) has all subsets of $\{1, 2, 3, 4\}$ with no consecutive elements, column (b) has binary strings of length 4 with no consecutive 1's, column (c) has lists of 1's and 2's with sum 5, column (d) has lists of odd integers with sum 6 and column (e) has lists of integers greater than 1 with sum 7.

There are several ways to show that the Fibonacci numbers give each of these counts. Two methods are:
Recursion - show that the objects being counted satisfy the same recurrence relation and initial conditions as the Fibonacci numbers.
Bijection - show a bijection, that is, a one to one correspondence, between the object and some other object known to be counted by the Fibonacci numbers

We will illustrate these methods with a few of the examples above but first we discuss bijections.

Bijections

A bijection between two sets is a one-to-one pairing of all of the elements in one set with all of the elements in the other set. If there is a bijection between two sets then they have the same cardinality. There are a few technical remarks: formally a bijection is a function from one set to another that is both onto (also called surjective) and one-to-one (also called injective). So everything in each set is paired to something in the other set. We have used the word cardinality rather than size to include infinite sets. For infinite sets the cardinality is in some sense a measure of how infinite it is. There is a bijection between the natural numbers and the rational numbers so both have the same cardinality (even though it might seem at first glance that there should be 'more' rational numbers). These are both countable sets as we can list or count them. It can be shown that there is no bijection between the natural numbers and the real numbers so the real numbers have a different cardinality. They are called uncountable. Infinite sets with the same cardinality as the natural numbers are countable those that do not have the same cardinality are uncountable. So, while the reals are uncountable, there are other uncountable sets with different cardinality than the reals.

The continuum hypothesis states that there is no set with cardinality between that of the natural numbers and the reals. Proving this was posed by Hilbert in 1900 as one of a set of famous problems. In 1963 Paul Cohen proved that the continuum hypothesis could neither be proved or disproved using a standard set of basic axioms. That is, it is undecidable in this setting.

Proofs for combinatorial description of Fibonacci numbers

We can think of subsets of $\{1, 2, \ldots, n\}$ in two ways. We can list the elements of the set or we can give a binary string of length n with 1's in locations corresponding to the elements of the set and 0's elsewhere. This binary list is called the characteristic vector. For example $\{2, 3, 5, 7\}$ as a subset of $\{1, 2, \ldots, 8\}$ is represented by 01101010. Technically we have just described a bijection between binary lists of length n and subsets of $\{1, 2, \ldots, n\}$. To formally prove that this is a bijection we would need to say a bit more but because this pairing is so clear we do not do so. Similarly, there is a bijection between subsets of $\{1, 2, \ldots, n-2\}$ with no consecutive elements and binary string of length $n-2$ with no consecutive 1's. Again we do not prove this formally. Note that by the bijection if one of these is counted by the Fibonacci numbers then so is the other.

To show that there are indeed F_n binary string of length $n-2$ with no consecutive 1's we could either show a bijection between these string and something else we know to be counted by the Fibonacci numbers or show that they satisfy the Fibonacci recurrence and have the same initial conditions. We will leave these as exercises.

To show that there are F_n lists of 1's and 2's with sum $n-1$ we will use the recurrence relation.

Proposition: The number of lists of 1's and 2's with sum $n-1$ is the Fibonacci number F_n.

Proof: Let R_n for $n \geq 1$ denote the set of all lists of 1's and 2's with sum $n-1$. Our aim is to show that $|R_n| = F_n$. Note that there is one list in R_1, the empty list and one list in R_2, the list with a single 1. So $|R_1| = 1 = F_1$ and $|R_2| = 1 = F_2$. Now, consider R_n and partition it into R_n^1 and R_n^2 where R_n^1 a contains those lists that end in a 1 and R_n^2 contains those lists that end in a 2. This is a partition as each list in R_n ends in exactly one of 1 or 2. So $|R_n| = |R_n^1| + |R_n^2|$.

Now, note that each element of R_n^1 is a 1,2 list with sum $n-1$ ending in 1. Dropping the last 1 results in a 1,2 list with sum $n-2$. Similarly, adding a 1 to the end of a 1,2 list with sum $n-2$ results in a list with sum $n-1$. This establishes a bijection between 1,2 lists with sum $n-1$ ending in 1 and 1,2 lists with sum $n-2$. Thus $|R_n^1| = |R_{n-1}|$. Similarly, dropping last 2 from lists in R_n^2 pairs them with lists in R_{n-2} and we have $|R_n^2| = |R_{n-2}|$. Thus we get that $|R_n| = |R_n^1| + |R_n^2| = |R_{n-1}| + |R_{n-2}|$. Thus the $|R_n|$ satisfy the Fibonacci recurrence and have the same initial conditions so $|R_n| = F_n$. $\square$

To show that the number of lists of odd positive integers with sum n and the number of lists of integers greater than 1 with sum $n+1$ are each F_n we could in a similar manner show that they satisfy the Fibonacci recurrence. In the first case we would partition the lists based on whether the last term is 1 or not. If it is 1 we drop the 1 to get a list with sum $n-1$. If it is not 1 it is at least 3 and we subtract 2 from the last term to get a list with sum $n-2$. We will not give a formal proof which would also need to show that the process described above is indeed a bijection, i.e., that it can be reversed. For the second case we would partition the lists based on whether the last term is 2 or not, dropping it if it is a 2 and subtracting 1 if it is not. Again we skip the details.

Having shown that the number of 1,2 lists is F_n we can also establish that the number of lists of odd positive integers with sum n and the number of lists of integers greater than 1 with sum $n+1$ are each F_n by establishing a bijections.

Consider again these lists when $n = 6$:

11111	111111	7
2111	3111	25
1211	1311	34
1121	1131	43
1112	1113	52
221	51	223
212	33	232
122	15	322

We need to find a way to pair the first column with each of the other two columns in a manner that will work in general for any n.

For the first two columns start first by adding a 1 to the front of each list in the first column so that each sum is now 6. Every consecutive string of 2's is now preceded by a 1. Replace each consecutive string of 2's and the preceding 1 by the sum of these terms. So, for example we would replace 12221 with 71 and replace 121 with 31. In the example for $n = 6$ this pairing put together entries in the same row. It is not hard to see that the sum is preserved and that we only get odd numbers so we do indeed get lists of odd positive integers with sum n. Also it is not hard to see that the process can be reversed, odd positive integers can be replaced with a list consisting of a 1 followed by some number of 2's. We will give a more formal description of this below.

For the first and third columns first start by adding a 2 to the front of each list in the first column so that each sum is now 7. Now each consecutive string of 1's is preceded by a 2. Replace each consecutive string of 1's and the preceding 2 by the sum of these terms. So, for example in 2212 we replace the middle 21 with a 3 to get 232 and in 21211 we replace the beginning 21 with 3 and the ending 211 with 4 to get 34. In the example for $n = 6$ this pairing put together entries in the same row. Again we can show that this produces a bijection in general but we will omit the details.

Now we give a more formal version of the proof for the first two columns:

Proposition: The number of lists of 1's and 2's with sum $n-1$ is the same as the number of lists of odd positive integers with sum n

Proof (using a bijection): Let S_n be the set of lists of 1's and 2's with sum $n-1$ and T_n be the set of lists of odd positive integers with sum n. We will give a bijection between S_n and T_n to establish the result.

Given a list in S_n, add a 1 to the start of the list. The new list has sum n and each consecutive string of 2's appearing in the new list is preceded by a 1. Replace each consecutive string of 2's and the preceding 1 by the sum of the entries. The sum must be odd and the only terms that are not changed are 1's that are followed by another 1. Thus the result is a list of odd positive integers with sum n. Reversing the process we can replace each odd integer $2k+1 \geq 3$ in a list in T_n with a string $122\ldots2$ having k 2's to get a list of 1's and 2's starting with a 1 having sum n. Dropping the first 1 gives a list in S_n. Each list in S_n then clearly yields a list in T_n and the reverse process takes the list in T_n back to the original list in S_n. Thus there is a bijection between S_n and T_n. $\square$

Binomial coefficients

In order to see another example in mathematics where Fibonacci numbers appear we first need to review binomial coefficients. Use the symbol $\binom{n}{k}$ to represent the number of k element subsets of an n element set. We read this as 'n choose k'. Although we do not need this fact now it is not hard to show that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ where $n!$ read as 'n factorial' is the product of all positive integers less than or equal to n. That is, $n! = n \cdot n-1 \cdot n-2 \cdots 3 \cdot 2 \cdot 1$. As a notational convenience we set $0!$ to be 1.

The following table, which we will call the binomial triangle gives some small values of $\binom{n}{k}$:

n \ k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

So, for example the entry in row 6 column 3 is $\binom{6}{3} = 20$. The number of different size 3 subsets of a 6 element set is 20. Note that we start counting at 0 so the row labeled 6 is actually the 7^{th} row in the table.

The binomial coefficients satisfy a simple recurrence relation that is useful for making the binomial triangle.

As a concrete example consider a class with 18 students where the instructor decides that exactly 7 students will get A's and the rest will get F's. The number of different groups (subsets) of 7 students that can be selected to get the A's is '18 choose 7', the binomial coefficient $\binom{18}{7}$. To get a sense of how large these number are, this is $\binom{18}{7} = \frac{18!}{7!11!} = 31,824$.

Imagine that the instructor has a favorite student. The size 7 subsets can be partitioned into two collections, those that contain the favorite student and those that do not. For those that contain the favorite, the instructor needs only to pick 6 students from the other 17 students, in $\binom{17}{6}$ ways. For those that do not contain the favorite, the instructor picks all 7 students from the remaining 17 students in $\binom{17}{7}$ ways. So we get $\binom{18}{7} = \binom{17}{6} + \binom{17}{7}$.

In general we get the following which we provide with a slightly more formal proof that follows the idea of previous paragraph.

Proposition: $\displaystyle\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof: Partition the k subsets of $\{1, 2, \ldots, n\}$ into two parts, the subsets containing n and the subsets not containing n. Each k subset of $\{1, 2, \ldots, n\}$ containing n can be written as $S \cup \{n\}$ for some $k-1$ subset of $\{1, 2, \ldots, n-1\}$ and hence the number of these is $\binom{n-1}{k-1}$. The k subsets of $\{1, 2, \ldots, n\}$ not containing n are the k subsets of $\{1, 2, \ldots, n-1\}$ and hence the number of these is $\binom{n-1}{k}$. The result then follows. $\square$.

This is sometimes called Pascal's identity and can be used to quickly determine entries in the binomial triangle. The right diagonal and first column entries are 1. Each other entry is the sum of the entry directly above it plus the entry above and one column to the left.

The binomial triangle is often presented as an isosceles triagle with the 1's forming the outer edges. It is often called Pascal's triangle in western countries, Yang Hui's triangle in China and Kayyam's triangle in Persia. 'Pascal's Triangle was discovered by Pingala around 500 A.D. in India, by Kayyam in Persia around 1000 A.D. and by Yang Hui in China around 1200 A.D. all long before Pascal's discovery in 1655. The binomial coefficients play an important role in various areas of mathematics. For example the rows give the coefficients in the expansion of $(x+y)^n$. This result plays a role, as one example, in determining probabilities related to sequences of coin flips.

The binomial triangle and Fibonacci numbers

Below we highlight two diagonal rows of the binomial triangle with italic and with bold numerals.

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	*1*					
4	1	4	*6*	4	**1**				
5	1	*5*	10	**10**	5	1			
6	*1*	6	**15**	20	15	6	1		
7	1	**7**	21	35	35	21	7	1	
8	**1**	8	28	56	70	56	28	8	1

The sum of the italic diagonal is $1+5+6+1 = 13 = F_7$, the sum of the bold diagonal is $1+7+15+10+1 = 34 = F_9$ and the sum of the diagonal between these two is $1+6+10+4 = 21 = F_8$. That these are Fibonacci numbers is no accident. The sums of the diagonals give the Fibonacci numbers.

In order to show this fact, we first need to get correct notation to express the observation of the previous paragraph and then we need to prove that it is correct. Writing the binomial coefficients the examples above are $F_7 = \binom{6}{0}+\binom{5}{1}+\binom{4}{2}+\binom{3}{3}$, $F_9 = \binom{8}{0}+\binom{7}{1}+\binom{6}{2}+\binom{5}{3}+\binom{4}{4}$ and $F_8 = \binom{7}{0}+\binom{6}{1}+\binom{5}{2}+\binom{4}{3}$. We see that if we start in row $n-1$ we get F_n so we guess that the Fibonacci-binomial identity is

$$F_n = \begin{cases} \binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\cdots+\binom{\frac{n-1}{2}}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \\ \binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\cdots+\binom{\frac{n}{2}}{\frac{n-2}{2}} & \text{if } n \text{ is even} \end{cases}$$

This is $F_n = \begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-2 \\ 1 \end{pmatrix} + \begin{pmatrix} n-3 \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} n-1-i \\ i \end{pmatrix} + \cdots$ where we understand that the sum stops when we get a term $\binom{a}{b}$ with $a < b$.

Proofs of the Fibonacci-binomial identity

We will give two different proofs of the Fibonacci- binomial identity described above in order to illustrate to different methods. One method, mathematical induction we have already encountered. The second method, a *combinatorial proof* we have also already encountered. The proof of the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ used an argument that involved counting size k subsets of n in two different ways. Equating the expression gave the identity. Using arguments that involve counting is the key to combinatorial proofs. Indeed, what we called a bijective proof above can also be considered a special type of combinatorial proof.

Proof (using mathematical induction): It is straightforward to check that the identity holds for $n = 1$ and $n = 2$: $F_1 = \binom{0}{0} = 1$ and $F_2 = \binom{1}{0} = 1$. By induction, we assume that the identity holds for values less than n and show that it holds for n. When n is even we get

$$\begin{array}{rcl}
F_{n-2} & = & \binom{(n-2)-1}{0} + \binom{(n-2)-2}{1} + \cdots + \binom{(n-2)-1-(i-1)}{i-1} + \cdots + \binom{\frac{(n-2)}{2}}{\frac{(n-2)-2}{2}} \\
+ \quad F_{n-1} & = & \binom{(n-1)-1}{0} + \binom{(n-1)-2}{1} + \binom{(n-1)-3}{2} + \cdots + \binom{(n-1)-1-i}{i} + \cdots + \binom{\frac{(n-1)-1}{2}}{\frac{(n-1)-1}{2}} \\
\hline
F_n & = & \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-1-i}{i} + \cdots + \binom{\frac{n}{2}}{\frac{n-2}{2}}
\end{array}$$

The equalities in the first two rows follow by induction. The bottom row is the sum of the top two rows and establishes the identity for F_n. The result on the left in the bottom follows by the Fibonacci recurrence and on the right from applying the binomial identity $\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$ to each pair of terms and noting that for the first term $\binom{n-2}{0} = 1 = \binom{n-1}{0}$. Similarly the identity can be shown for F_n when n is odd as follows:

$$\begin{array}{rcl}
F_{n-2} & = & \binom{(n-2)-1}{0} + \cdots + \binom{(n-2)-1-(i-1)}{i-1} + \cdots + \binom{\frac{(n-2)+1}{2}}{\frac{(n-2)-3}{2}} + \binom{\frac{(n-2)-1}{2}}{\frac{(n-2)-1}{2}} \\
+ \quad F_{n-1} & = & \binom{(n-1)-1}{0} + \binom{(n-1)-2}{1} + \cdots + \binom{(n-1)-1-i}{i} + \cdots + \binom{\frac{(n-1)}{2}}{\frac{(n-1)-2}{2}} \\
\hline
F_n & = & \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-1-i}{i} + \cdots + \binom{\frac{n+1}{2}}{\frac{n-3}{2}} + \binom{\frac{n-1}{2}}{\frac{n-1}{2}}
\end{array}$$

The equalities in the first two rows follow by induction. The bottom row is the sum of the top two rows and establishes the identity for F_n. The result on the left in the bottom follows by the Fibonacci recurrence and on the right from applying the binomial identity $\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$ to each pair of terms and noting that for the first term $\binom{n-2}{0} = 1 = \binom{n-1}{0}$ and for the last term $\binom{\frac{n-3}{2}}{\frac{n-3}{2}} = 1 = \binom{\frac{n-1}{2}}{\frac{n-1}{2}}$.

Thus by induction the identity holds for all n. $\square$

Proof (using combinatorial methods): The number of lists of 1's and 2's with sum $n-1$ is F_n. Partition these lists into $S_0, S_1, S_2 \ldots$, where S_i consists of those lists with i terms that are 2's and hence $(n-1) - 2i$ terms that are 1's. Since the S_i partition the lists (i.e., the S_i are disjoint and their union is all of the lists) we have that $F_n = |S_0| + |S_1| + \cdots$. The lists in S_i have length $(n-1-2i) + i = n-1-i$ of which i are twos. As there are $\binom{n-1-i}{i}$ ways to the locations of the 2's we get that $|S_i| = \binom{n-1-i}{i}$. Hence $F_n = |S_0| + |S_1| + \cdots = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-k}{k-1} + \cdots$ establishing the identity. $\square$

Another Fibonacci identity

Another identity involving Fibonacci numbers is

$$1 + F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2}.$$

We will outline a proof by mathematical induction, a combinatorial proof and also discuss using $\sum$ notation.

Mathematical induction

Learning mathematical induction may be a bit like learning to ride a bike. It seems almost impossible at first and you fall and scrape your knees a few times but eventually you get it and it turns out that it isn't hard and you can really do a lot once you figure it out.

To get to the ideas behind a mathematical induction proof we describe the process in detail. The final proof will hide all of this and be very short.

Substituting $n = 0$ into the equation we get $1 + F_0 = F_2$. Since $F_0 = 0$ and $F_2 = 1$ this is $1 + 0 = 1$ and clearly true. When $n = 1$ we get $1 + F_0 + F_1 = F_3$. With $F_0 = 0.F_1 = 1$ and $F_3 = 2$ we have $1 + 0 + 1 = 2$ and the statement is true. Using the recurrence for Fibonacci numbers we have $F_3 = F_2 + F_1$ and substituting $F_2 = 1 + F_0$ which we have just established we get $F_3 = F_2 + F_1 = (1 + F_0) + F_1$ as needed.

For $n = 2$ we need to establish $F_4 = 1 + F_0 + F_1 + F_2$. Using the Fibonacci recurrence and $F_3 = 1+F_0+F_1$ which we have just shown to be true we get $F_4 = F_3+F_2 = (1+F_0+F_1)+F_2$ and the identity holds for $n = 2$.

Now imagine that we keep doing this and have shown the identity for $n = 0, 1, 2, \ldots, 11$ and we want to show it for $n = 12$. That is, we already know the following:

$$\begin{array}{rcl} F_2 & = & 1 + F_0 \\ F_3 & = & 1 + F_0 + F_1 \\ F_4 & = & 1 + F_0 + F_1 + F_2 \\ \vdots & \vdots & \vdots \\ F_{13} & = & 1 + F_0 + F_1 + \cdots + F_{11} \end{array}$$

We want to know show that $F_{14} = 1 + F_0 + F_1 + \cdots + F_{11} + F_{12}$. Substituting the last line above (which we have already established to be true) and the recurrence we get

$$F_{14} = F_{13} + F_{12} = (1 + F_0 + F_1 + \cdots + F_{11}) + F_{12} = 1 + F_0 + F_1 + \cdots + F_{12}$$

So using the fact that the identity is known to be true for $n = 0, 1, \ldots, 11$ we have shown that it is true for $n = 12$. In fact we really only needed the fact that the identity is known to be true for $n = 11$. That it is true for $n = 0, 1, \ldots, 10$ is in this case extra unused information. (In some induction proofs we will need the truth of the statement for all smaller values.)

There was nothing to special about $n = 11$ so just use the symbol n. That is, we imagine that we have shown the identity true for $1, 2, \ldots, (n-1)$ and we want to show that it is true for n. So we assume that we already know the following:

$$\begin{array}{rcl} F_2 & = & 1 + F_0 \\ F_3 & = & 1 + F_0 + F_1 \\ F_4 & = & 1 + F_0 + F_1 + F_2 \\ \vdots & \vdots & \vdots \\ F_{(n-1)+2} & = & 1 + F_0 + F_1 + \cdots + F_{n-1} \end{array}$$

We want to use these fact and anything else we know (in this case the Fibonacci recurrence) to show the identity for n. That is we need to show $F_{n+2} = 1 + F_0 + F_1 + \cdots + F_n$ using the assumptions above.

We can do this fairly easily by substituting the last line of the values assumed to be true, where the left side is $F_{(n-1)+2} = F_{n+1}$ and the recurrence to get

$$F_{n+2} = F_{n+1} + F_n = (1 + F_0 + \cdots + F_{n-1}) + F_n = 1 + F_0 + \cdots + F_{n-1} + F_n.$$

Observe that the second equality is simply using the associative property. We didn't really need to do this but include it to make sure things are clear.

So we have shown that the identity holds at n when it is assumed to be true for $1, 2, \ldots, n-1$. This we can imagine a process of building up values for which we know it is true, $n = 0$ then $n = 1$ then $n = 2$ then $\cdots$ then $n = 11$ then $n = 12$ $\cdots$. Clearly we can do this to get all numbers $0, 1, 2, \ldots$ and this established that the identity is true for all n.

This is mathematical induction. The wording of a proof using mathematical induction is a way of describing the process we went through above without writing so much. If we all understand that this is what is meant by induction then we can present the proof ideas very succinctly.

Here is how we do this using induction:

Proof that $1 + F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2}$ *by mathematical induction. (version 1)*

Since $F_0 = 0$ and $F_2 = 1$ we have $F_2 = 1 = 1 + 0 = 1 + F_0$. So the identity holds for $n = 0$. For $n \geq 1$ by the Fibonacci recurrence we have $F_{n+2} = F_{n+1} + F_n$ and by induction we can assume $F_{n+1} = 1 + F_0 + \cdots + F_{n-1}$. Then $F_{n+2} = F_{n+1} + F_n = (1 + F_0 + \cdots + F_{n-1}) + F_n$ and the identity holds for n. Thus by mathematical induction $1 + F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2}$ for $n = 0, 1, 2, \ldots$. $\square$

$\sum$ notation

We next introduce some more notation for the proof above. In this case the $\sum$ notation does not do too much to make the proof above shorter or more clear but in many instances it does so it is good practice to use the notation here.

We use $\sum$ as shorthand for sums. So $\displaystyle\sum_{i=0}^{n} F_i$ means that we sum all of the expressions that we get substituting values $i = 1, 2, \ldots, n$ into the expression after the $\sum$. That is, $\displaystyle\sum_{i=0}^{n} F_i = F_0 + F_1 + \cdots + F_n$.

Now we give the proof above again using this notation. We also phrase it even slightly more succinctly to see how induction proofs are sometimes written. By the basis we are referring to the small case $n = 0$ that is checked directly.

Proof that $1 + \sum_{i=0}^{n} F_i = F_{n+2}$ *by mathematical induction. (version 2)*

The basis is $F_2 = 1 = 1 + 0 = 1 + F_0$. For $n \geq 1$ we have

$$F_{n+2} = F_{n+1} + F_n = (1 + \sum_{i=0}^{n-1} F_i) + F_n = 1 + \sum_{i=0}^{n} F_i$$

where the first equality is from the Fibonacci recurrence, the second by induction and the last by including F_n in the sum. So, by induction the identity holds for all n. $\square$

Combinatorial proof

We also want to show

$$1 + F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2}$$

using a counting argument. This is what is known as a combinatorial proof. In this instance the combinatorial proof is at least as long as the induction proof. However, combinatorial proofs are in general a powerful technique that sometimes are much shorter than other methods. They also are 'nice' in that they do not require as much 'overhead' such as, for example, having a good grasp on the technique of mathematical induction.

We recall that the Fibonacci numbers count the number of lists of 1's and 2's with sum $n-1$. When $n = 5$ and so $n + 2 = 7$ we have the following $F_7 = 13$ lists of 1's and 2's with sum 6. We will soon see why they are arranged in the columns as done here.

111111		21111	12111	11211	11121	11112
				2211	2121	2112
					1221	1212
						1122
						222

This is almost too small to see a pattern but we can see something. Observe that the lists in the last column all end in 2, those in the second to last column end in 21 etc. That is, we have partition the lists based on how many 1's follow the last 2. The first column has the only list with seven 1's and no 2's. The second column has no lists since there are no lists with sum 6 ending with a 2 followed by six 1's (the sum would be too big), the third column ends with a 2 followed by five 1's etc. Consider the column with those lists that end in 21. The beginning of the list, preceding the end 21 has sum $6 - 3 = 3$ and we know that there are $F_4 = 3$ such lists. So this column should have size 3.

For a given n the right side of the identity is F_{n+2} which counts lists of 1's and 2's with sum $(n + 2) - 1 = n + 1$. The goal will be to partition the set of all lists with sum $n + 1$ into subsets of the lists with sizes corresponding to the sizes given by Fibonacci numbers on the left side of the identity.

For any n we can partition the set of F_{n+2} lists of 1's and 2's with sum $n + 1$ based on the number of 1's following the last 2. Observe that lists with sum $n+1$ that end with $211\ldots1$ where there are k terms that are 1 can be obtained by taking lists with sum $(n+1)-2-k = (n - k) - 1$ and appending $211\ldots1$, with sum $k + 2$. So there are F_{n-k} such lists. It seems that this partition the gives the count of F_{n+2} as a sum of smaller Fibonacci numbers, plus 1 for the single list with all 1's.

What we need to do now is take the observation of the previous paragraph and make it into a proof that works for all n. We give two versions of the same proof using slightly different notation and level of formality.

Combinatorial proof that $1 + F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2}$. *(version 1)*

Let S be the set of lists of 1's and 2's with sum $n+1$. This has size F_{n+2}. Partition S into $(T_0 \cup T_1 \cup \cdots T_{n-1}) \cup U$ where U consists of the single list with $n+1$ terms that are 1 and T_k consists of all lists for which the last 2 is followed by exactly k terms that are 1's. This is a partition. Each list in S is in exactly one of these sets as any list with sum $n+1$ that contains a 2 has at most $n-1$ terms that are 1. So $|S| = |T_0| + |T_1| + \cdots + |T_{n-1}| + |U|$. Given a list in T_k, deleting the last 2 and the 1's that follow it results in a list with sum $(n+1) - 2 - k = (n-k) - 1$. Reversing this, adding a 2 followed by k terms that are 1's to a list with sum $(n-k)-1$ yields a list in T_k. Thus there is a bijection between T_k and lists of 1's and 2's with sum $(n-k)-1$. So $|T_k| = F_{n-k}$. Along with $|U|$ by its definition and using $F_0 = 0$ we get

$$\begin{aligned}
F_{n+2} &= |S| \\
&= |U| + 0 + |T_{n-1}| + |T_{n-2}| + \cdots + |T_0| \\
&= 1 + F_0 + F_{n-(n-1)} + F_{n-(n-2)} + \cdots + F_{n-0} \\
&= 1 + F_0 + F_1 + F_2 + \cdots + F_n
\end{aligned}$$

and the identity is shown. $\Box$

Combinatorial proof that $1 + \sum_{i=0}^{n} F_i = F_{n+2}$. *(version 2)*

There are F_{n+2} lists of 1's and 2's with sum $n+1$. There is one list of all 1's. The remaining lists each contain at least one 2. There are F_{n-k} lists in which the last 2 is followed by k terms that are 1 because such lists are determined by the part of the list preceding the last 2, which has sum $(n+1) - 2 - k = (n-k) - 1$. here k can be $0, 1, \ldots, n-1$. Thus

$$F_{n+2} = 1 + \sum_{k=0}^{n-1} F_{n-k} = 1 + \sum_{i=1}^{n} F_i = 1 + \sum_{i=1}^{n} F_i$$

where the last equality follows by adding $F_0 = 0$ and the second to last from the change of indices $i = n - k$. So the identity is shown. $\Box$