Math 163 - Introductory Seminar: Combinatorial Duality

Spring 2008 - Lehigh University (Contact Garth Isaak, gisaak@lehigh.edu for more information.)

The mathematics department offers Math 163 as 'an introduction to the discipline of mathematics designed for students considering a major in mathematics. The course will provide an introduction to rigorous mathematical reasoning and will survey some area of mathematics. Topics covered will vary.'

That is, we pick some interesting topic to study in detail with the idea that students get a gentle introduction to the abstract reasoning and theorem proving that is found in many upper level mathematics courses. This is a good first course for students beginning the process of becoming proficient in this sort of reasoning. It is required for the major and it is recommended that it be taken early. However it is open to any students, including non-majors who might be considering a mathematics major or minor or students with other majors who enjoy mathematics.

The topic for this spring is 'combinatorial duality.' The description here is meant to give a hint of what that means.

The pair of equations $\begin{array}{ccccccc} 2x & + & 3y & + & 4z & = & 3 \\ x & - & y & - & 2z & = & -2 \end{array}$ does not have a non-negative solution. To see this multiply the second by 2 and add to see that $4x + y = -1$ for any solution. Geometrically, the line that is the intersection of the two corresponding planes does not contain any non-negative points. Alternatively, the cone formed by vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ does not contain $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. We will discuss the general theory behind this observation, looking at it geometrically, using linear algebra and considering some of the mathematics related to algorithms that solve such problems. (If you have taken linear algebra this may look a bit like what you did, if you haven't, don't worry as we will start from the basics.) Solving such problems in large scale is a practical tool used extensively by many businesses for efficient resource use. (We will however be more interested in the underlying mathematics.) If you add the condition that the solution must be integral then things are more complicated and we can only get nice theorems in certain situations. (Indeed, you would win one of the million dollar Clay prizes if you can prove that it is impossible to devise an 'efficient' algorithm to solve such problems with integral conditions in general.)

We will then look at some interesting combinatorial theorems and consider methods of directly proving them as well as seeing how they follow from the results described above, viewing the matrix results as a framework for a general set of 'combinatorial duality' results. A few of these problems that can be easily described here are as follows (there are many others):
- We repeatedly play a game where we each show a dime or a quarter, if they match you give me your coin, otherwise I give you mine. My best strategy is to randomly play dimes $5/7$ of the time and quarters $2/7$ of the time and your is to randomly play both coins with equal probability.
- All of the boys at a heterosexual dance can be paired with girls if and only if for any set of k boys one can find at least k girls who like at least one of the boys in the set.
- The maximum flow that can go through a set of pipelines from Denver to Boston is equal to the minimum capacity of the pipes crossing a line cutting Denver off from Boston
- A largest collection of size $k \leq n/2$ subsets of $\{1, 2, \ldots, n\}$ with each pair of subsets intersecting is obtained by taking all subsets containing 1.
- A list of numbers corresponds to the list of numbers of wins for teams in some round robin tournament if and only if the sum of each subset of the numbers (wins) is at least as large as the number of games played by the corresponding teams.