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Proper and unit bitolerance orders and graphs

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Abstract

We say any order $\prec$ is a *tolerance order* on a set of vertices if we may assign to each vertex x an interval I_x of real numbers and a real number t_x called a tolerance in such a way that $x \prec y$ if and only if the overlap of I_x and I_y is less than the minimum of t_x and t_y and the center of I_x is less than the center of I_y. An order is a *bitolerance order* if and only if there are intervals I_x and real numbers $t_l(x)$ and $t_r(x)$ assigned to each vertex x in such a way that $x \prec y$ if and only if the overlap of I_x and I_y is less than both $t_r(x)$ and $t_l(y)$ and the center of I_x is less than the center of I_y. A tolerance or bitolerance order is said to be bounded if no tolerance is larger than the length of the corresponding interval. A *bounded tolerance graph* or *bitolerance graph* (also known as a *trapezoid graph*) is the incomparability graph of a bounded tolerance order or bitolerance order. Such a graph or order is called *proper* if it has a representation using intervals no one of which is a proper subset of another, and it is called *unit* if it has a representation using only unit intervals. In a recent paper, Bogart, Fishburn, Isaak and Langley (1995) gave an example of proper tolerance graphs that are not unit tolerance graphs. In this paper we show that a bitolerance graph or order is proper if and only if it is unit. For contrast, we give a new view of the construction of Bogart et al. (1995) from an order theoretic point of view, showing how linear programming may be used to help construct proper but not unit tolerance orders.

1. Introduction

An order $\prec$ on a set X is called an *interval order* if it is possible to assign to its vertices x intervals I_x of real numbers is such a way that $x \prec y$ if and only if all members of I_x are less than all members of I_y. We will discuss only orders with $|X|$ finite. An order is a *(min) tolerance (interval) order* on a set of vertices if we may assign to each vertex x an interval I_x of real numbers and a real number t_x called a tolerance in such a way that $x \prec y$ if and only if the overlap of I_x and I_y is less than the minimum of t_x and t_y and the center of I_x is less than the center of I_y. An order is a *(min)*

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bitolerance (*interval*) *order* if and only if there are intervals I_x and real numbers $t_l(x)$ and $t_r(x)$ assigned to each vertex x in such a way that $x \prec y$ if and only if the overlap of I_x and I_y is less than both $t_r(x)$ and $t_l(y)$ and the center of I_x is less than the center of I_y. The assignment of intervals and tolerances is called a *representation* of the order. A tolerance or bitolerance order is called *bounded* if each tolerance is no more than the corresponding interval length. All known examples of tolerance and bitolerance orders are bounded. Bounded bitolerance orders turn out to be exactly the orders of interval dimension two [3]; the proof of Theorem 1 in this paper suggests why this is so. An *interval graph*, *bounded* (*min*) *tolerance* (*interval*) *graph*, or *bounded* (*min*) *bitolerance* (*interval*) *graph* (also known as a *trapezoid graph* [5]) is the incomparability graph of an interval order, a bounded tolerance order, or a bounded bitolerance order. Such a graph or order is called *proper* if it has a representation using intervals no one of which is a proper subset of another, and it is called *unit* if it has a representation using only unit intervals. (Unit interval orders are also known as *semiorders* [16] and unit interval graphs are also known as *indifference graphs* [18].) Tolerance graphs were introduced by Golumbic and Monma [11] (as above, we assign intervals and tolerances to vertices and two vertices are adjacent if their intervals overlap by at least one of their tolerances) and studied in more detail by Golumbic et al. [12]. This paper asks whether a tolerance graph which has an orientable complement is necessarily bounded; this question is as yet unanswered. Tolerance graphs were put into a more general format by Jacobson et al. [13]. For simplicity of language, we henceforth omit the optional min and interval when we refer to tolerance or bitolerance graphs or orders. It is straightforward to show that a proper tolerance graph (or order) is bounded, so the adjective bounded may be omitted when referring to bounded proper tolerance graphs (or orders) or bounded unit tolerance graphs or (orders).

Fishburn [10, 8, 2] showed that an ordering of X is an interval ordering if and only if it has no four element restriction isomorphic to the ordering $\underline{2} + \underline{2}$ illustrated by its covering diagram in Fig. 1, consisting of vertices a, b, c, and d, with $a < b$, $c < d$, and no other relations. Scott and Suppes [19] showed that an ordering is a unit interval ordering if and only if it has no four element restriction isomorphic to $\underline{2} + \underline{2}$ and no four element restriction isomorphic to the ordering $\underline{3} + \underline{1}$, also illustrated by its covering diagram in Fig. 1, consisting of four vertices a, b, c, and d, with $a < b < c$, $a < c$, and d incomparable with all of a, b, and c. A graph is thus an interval graph if and only if it is a co-comparability graph (the complement of a transitively orientable graph) and has no induced subgraph isomorphic to a four-cycle (the incomparability graph of $\underline{2} + \underline{2}$). The similar characterization of unit interval graphs [18] requires in addition that they have no induced subgraph isomorphic to the 'claw' $K_{1,3}$, the complete bipartite graph with parts of size 1 and 3. There are characterizations of interval graphs and unit interval graphs that do not include the hypothesis that the graph is a cocomparability graph; see e.g., [21].

It is clear that a unit interval order (graph) is proper. Roberts first noted that a proper interval graph (order) is unit. Since a proper interval order contains no

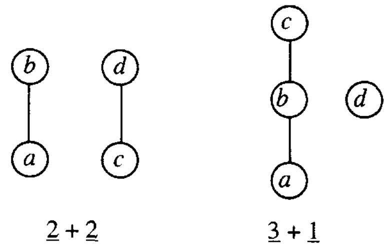

Fig. 1.

restriction isomorphic to $\underline{3} + \underline{1}$, this result is an immediate corollary of the Scott–Suppes theorem [19] in the order theoretic context. It is also clear that unit (bi)tolerance orders are proper. Thus it is natural to ask whether proper (bi)tolerance orders are unit. Perhaps surprisingly, the answer is yes for bitolerance orders and no for tolerance orders.

2. Proper and unit bitolerance orders

Associated with any bitolerance order on X, there is a natural linear extension of that ordering: given a representation (in which we may assume, without loss of generality, no two intervals I_x and I_y have the same center), we define the linear extension by $x <_c y$ if and only if the center of x is less than the center of y. We refer to this linear ordering as the *central extension* of the representation and denote it with $<_c$. Our first two lemmas show that the possible central extensions of a proper bitolerance order are quite restricted.

Lemma 1. *Let N be a four element restriction of a proper bitolerance ordering to four elements a, b_1, b_2, and d with $a \succ b_1$, $b_1 \prec b_2$, $b_2 \succ d$ and with no other comparabilities among a, b_1, b_2 and d. Then in any central extension of the ordering, either $b_1 <_c d <_c b_2$ or $b_1 <_c a <_c b_2$ (or both).*

Proof. Suppose there is a central extension in which $d <_c b_1$ and $a >_c b_2$. Then, as illustrated in Fig. 2,

$$|I_a \cap I_d| \leqslant |I_{b_2} \cap I_d| < \min\{t_r(d), t_l(b_2)\} \leqslant t_r(d)$$

since $b_2 \succ d$ and

$$|I_a \cap I_d| \leqslant |I_{b_1} \cap I_a| < \min\{t_r(b_1), t_l(a)\} \leqslant t_l(a)$$

since $a \succ b_1$. So $d \prec a$, contrary to hypothesis. □

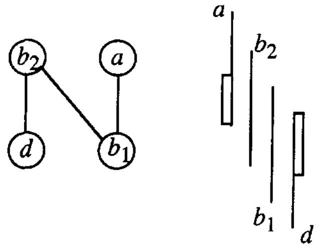

Fig. 2. The rectangles represent the left tolerance of a and the right tolerance of d.

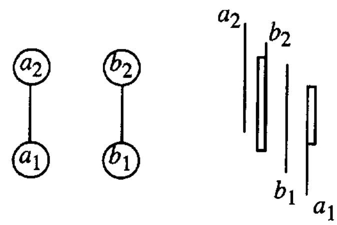

Fig. 3. The rectangles represent the left tolerance of b_2 and the right tolerance of a_1.

Lemma 2. *Let* $\underline{2} + \underline{2}$ *be a four element restriction of a proper bitolerance ordering to four elements* a_1, a_2, b_1, *and* b_2 *with* $a_1 \prec a_2$, $b_1 \prec b_2$ *and no other comparabilities among* a_1, a_2, b_1, *and* b_2. *Then in any central extension of the ordering, either* $a_1 <_c b_1 <_c b_2 <_c a_2$ *or* $b_1 <_c a_1 <_c a_2 <_c b_2$.

Proof. We may assume without loss of generality that $a_1 <_c b_1$. Then, as in Fig. 3,

$$|I_{a_1} \cap I_{b_2}| \leqslant |I_{b_1} \cap I_{b_2}| < \min\{t_r(b_1), t_l(b_2)\} \leqslant t_l(b_2),$$

since $b_1 \prec b_2$. Then since $a_1 \not\prec b_2$,

$$|I_{a_1} \cap I_{b_2}| \geqslant \min\{t_r(a_1), t_l(b_2)\}$$

so that

$$|I_{a_1} \cap I_{b_2} \geqslant t_r(a_1).$$

Since $a_1 \prec a_2$,

$$|I_{a_1} \cap I_{a_2}| < \min\{t_r(a_1), t_l(a_2)\} \leqslant t_r(a_1);$$

therefore the left endpoint of I_{a_2} must be greater than that of I_{b_2}, and since the representation is proper, the centers must be in that order too. Thus $a_1 <_c b_1 <_c b_2 <_c a_2$. □

Lemma 2 appears in graph theoretic terminology (and slightly less generality) as Lemma 5 of [1]. The required linear extensions of the restrictions in Lemmas 1 and 2 characterize proper and unit bitolerance graphs in the following sense.

Theorem 1. *For an order* $\prec$ *on a set* X *the following are equivalent*:
1. $(X, \prec)$ *is a proper bitolerance ordering*;
2. *The order* $\prec$ *has a linear extension* $<_L$ *such that*
   (a) *if* a, b_1, b_2, *and* d *are elements of* X *with* $a \succ b_1$, $b_1 \prec b_2$, $b_2 \succ d$ *and no other comparabilities among these elements, then in* L, *either* $b_1 <_L d <_L b_2$ *or* $b_1 <_L a <_L b_2$ *(or both), and*

(b) *if* a_1, a_2, b_1, *and* b_2 *are elements of* X *with* $a_1 \prec a_2$ *and* $b_1 \prec b_2$ *and no other comparabilities among* a_1, a_2, b_1, *and* b_2, *then in* L, *either* $a_1 <_L b_1 <_L b_2 <_L a_2$ *or* $b_1 <_L a_1 <_L a_2 <_L b_2$.

3. $(X, \prec)$ *is a unit bitolerance ordering.*

Proof. Statement 3 implies statement 1 just as for interval orderings. Lemmas 1 and 2 show that statement 1 implies statement 2. Thus we assume that statement 2 holds. We introduce a bit more notation to ease the proof of statement 3. We denote the left endpoint and right endpoint of the interval I_x by $\mathrm{le}(x)$ and $\mathrm{re}(x)$, respectively. We define the *left tolerant point* of I_x by

$$\mathrm{lt}(x) = \mathrm{le}(x) + t_l(x)$$

and the *right tolerant point* of I_x by

$$\mathrm{rt}(x) = \mathrm{re}(x) - t_r(x).$$

Then it is immediate from the definitions that $x \prec y$ if and only if,

$$\mathrm{rt}(y) < \mathrm{le}(y) \quad \text{and} \quad \mathrm{re}(x) < \mathrm{lt}(y).$$

(Note that this shows that $x \prec y$ if and only if all points of the real interval $[\mathrm{le}(x), \mathrm{rt}(x)]$ are to the left of all the points of the real interval $[\mathrm{le}(y), \mathrm{rt}(y)]$ and all points of $[\mathrm{lt}(y), \mathrm{re}(x)]$ are to the left of all points of the real interval $[\mathrm{lt}(y), \mathrm{re}(y)]$; those familiar with the concept of interval dimension [21] will note this implies bounded bitolerance orders have interval dimension two; the converse is true as well.)

Now suppose the linear ordering of statement 2 is $x_1 <_L x_2 <_L \cdots <_L x_n$. We choose real numbers

$$\mathrm{le}(x_1) < \mathrm{le}(x_2) < \cdots < \mathrm{le}(x_n).$$

Next we choose a real number $\mathrm{re}(x_1)$ greater than $\mathrm{le}(x_n)$, and we let u (for unit) equal $\mathrm{re}(x_1) - \mathrm{le}(x_1)$. Now we define the remaining right endpoints by

$$\mathrm{re}(x_i) = \mathrm{le}(x_i) + u.$$

Finally, we define the left and right tolerant points. First, if $x_i \not\prec x_j$ for any j, define

$$\mathrm{rt}(x_i) = \tfrac{1}{2}\,[\mathrm{le}(x_n) + \mathrm{re}(x_1)].$$

Then $\mathrm{rt}(x_i)$ is in each interval I_{x_j}, so $x_i \not\prec x_j$ in this representation for $j > i$. (For $j < i$, the center of x_j is less than the center of x_i so $x_i \not\prec x_j$ in this representation.) Otherwise, let a be the smallest integer such that $x_i \prec x_a$ (x_a is *above* x_i),

$$\mathrm{rt}(x_i) = \tfrac{1}{2}\,[\mathrm{le}(x_a) + \mathrm{le}(x_{a-1})]$$

Similarly, if $x_j \not\succ x_i$ for any i, define

$$\mathrm{lt}(x_j) = \tfrac{1}{2}\,[\mathrm{le}(x_n) + \mathrm{re}(x_1)].$$

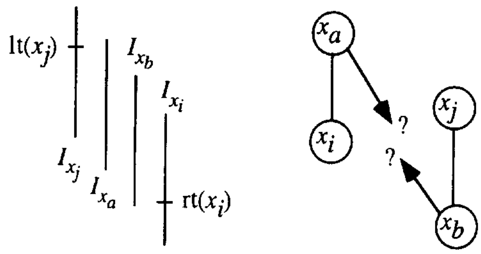

Fig. 4.

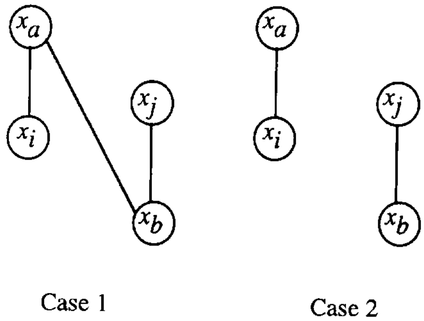

Fig. 5.

Otherwise, let b be the largest integer such that $x_b \prec x_j$ (x_b is *below* x_j), and let

$$\mathrm{lt}(x_j) = \tfrac{1}{2}\,[\mathrm{re}(x_b) + \mathrm{re}(x_{b+1})].$$

By our choices, if $x_i \prec x_j$, then $\mathrm{rt}(x_i) < \mathrm{le}(x_a) \leqslant \mathrm{le}(x_j)$ and $\mathrm{re}(x_i) \leqslant \mathrm{re}(x_b) < \mathrm{lt}(x_j)$. So the representation is consistent with $(X,\prec)$ in these cases.

If x_i and x_j (with $i<j$) are incomparable in $(X,\prec)$, we will assume that the representation gives $x_i \prec x_j$ and reach a contradiction. If the representation gives $x_i \prec x_j$ then $\mathrm{rt}(x_i) < \mathrm{le}(x_j)$ and $\mathrm{rl}(x_i) < \mathrm{lt}(x_j)$. Then, with $\mathrm{rt}(x_i) < \mathrm{le}(x_j)$ and a as defined above, $\mathrm{le}(x_j) > \mathrm{le}(x_a)$. Since $\mathrm{re}(x_i) < \mathrm{lt}(x_j)$, this means that, with b as defined above, $\mathrm{re}(x_i) < \mathrm{re}(x_b)$. As the illustration in Fig. 4 shows, this means that $x_a <_L x_j$ and $x_i <_L x_b$.

While we know the relationship between x_i and x_j in $(X,\prec)$, we do not know the relationships of x_a and x_b with x_j and x_i respectively and with each other in $(X,\prec)$. However if $x_b \succ x_i$, then by transitivity $x_j \succ x_i$, which is false. Further, if $x_b \prec x_i$, then $x_b <_L x_i$, contradicting the inequality $x_i <_L x_b$ above. Thus x_b is incomparable to x_i and similarly, x_a is incomparable to x_j. This leaves us two possible orderings among x_i, x_j, x_a, and x_b, as illustrated in Fig. 5.

In case 1, part (a) of statement 2 of the theorem tells us that either $x_b <_L x_j <_L x_a$ or $x_b <_L x_i <_L x_a$, contradicting one of $x_a <_L x_j$ and $x_i <_L x_b$ above. In case 2, part (b) of statement 2 of the theorem tells us that either $x_i <_L x_b <_L x_j <_L x_a$ or $x_b <_L x_i <_L x_a <_L x_j$, again contradicting $x_a <_L x_j$ and $x_i <_L x_b$ from above. Thus x_i and x_j must be incomparable in this representation if they are incomparable in P. □

3. The Fishburn model

Peter Fishburn [9] suggested that the following model of an ordered set would prove interesting for study. We choose, for each x in a set X, an interval I_x and a point c_x in I_x. We define $x \prec y$ if and only if all the elements of I_x are less than c_y and all elements of I_y are greater than c_x. Thus our ordering has a tolerance representation in which the sum of the tolerances $\mathrm{lt}(x)$ and $\mathrm{rt}(x)$ is the interval length $|I_x|$ for each element x. We call such a representation a *Fishburn representation*. If an order has a Fishburn representation, then its incomparability graph is a pseudo-interval graph in the sense of [4] and has a natural orientation as an interval catch digraph [17]. Thus an alternate name for a Fishburn representation is an *interval catch representation*.

Fishburn was interested in Fishburn representations with the additional condition that the intervals are all unit intervals; he observed that in this case the orders defined here have the fascinating property that they have no restrictions isomorphic to $\underline{2} + \underline{3}$ or $\underline{4} + \underline{1}$. Our interest is in the more general representation.

Our next theorem appears in [15] (with a geometric proof) as part of Theorem 2.5. Notice that the theorem does not say that a proper or unit bitolerance ordering has a Fishburn representation with unit intervals; the interval bitolerance representation which is unit need not have the sum of the left and right tolerances equal to the interval length.

Theorem 2. *An ordering is a proper (unit) bitolerance ordering if and only if it has a Fishburn representation.*

Proof. Suppose we have a unit bitolerance representation of an ordering $(X,\prec)$ with intervals I_x of length u and left and right tolerance $t_l(x)$ and $t_r(x)$. We will change the lengths of the intervals to $2u - t_l(x) - t_r(x)$. In particular, if $I_x = [\mathrm{le}(x), \mathrm{re}(x)]$, then we let

$$I'_x = [\mathrm{le}(x) + t_l(x) - u/2,\ \mathrm{re}(x) - t_r(x) + u/2] = [\mathrm{le}'(x), \mathrm{re}'(x)],$$

and we let

$$c_x = \tfrac{1}{2}\,[\mathrm{le}(x) + \mathrm{re}(x)].$$

The reason for these choices is that if $c_x < c_y$, then I_x overlaps I_y by at least $t_r(x)$ if and only if, when we increase the right endpoint of I_x by $u/2$ and decrease the left endpoint by $u/2$, the new (intermediate) intervals, which we denote by I^*_x and I^*_y, that we get overlap by $u + t_r(x)$. However these intermediate intervals overlap by $u + t_r(x)$ if and only if when we decrease the right endpoint of I^*_x by $t_r(x)$ and increase the left endpoint of I^*_y by $t_l(y)$, the resulting intervals I'_x and I'_y overlap by at least $u - t_l(y)$. However, $u - t_l(y)$ is the distance from $\mathrm{le}'(y) = \mathrm{le}(y) + t_l(y) - u/2$ to $\frac{1}{2}[\mathrm{le}(y) + \mathrm{re}(y)]$, i.e. the distance from $\mathrm{le}'(y)$ to c_y. Thus I_x and I_y overlap by at least $t_r(x)$ if and only if I'_x contains c_y. Similarly, I_x and I_y overlap by at least $t_l(y)$ if and only if I'_y contains c_x. This gives Fishburn representation of $(X,\prec)$. Given a Fishburn representation, we can reverse the construction with any sufficiently large u. □

As a corollary we obtain a theorem found in [1].

Corollary 1. *An order is a unit tolerance order if and only if it is a 50% tolerance order, i.e. it has a tolerance representation in which the tolerance of each x is half the length of I_x.*

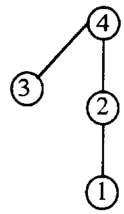

Fig. 6.

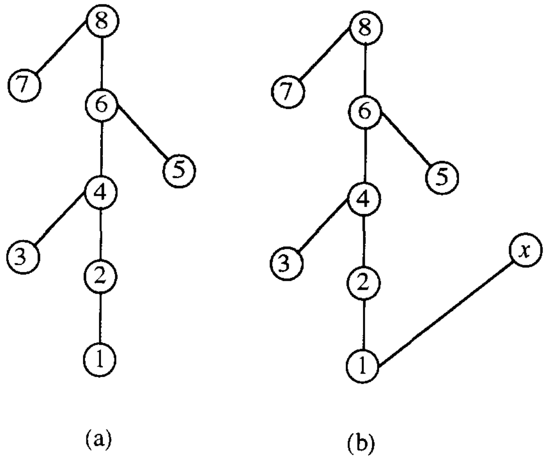

Fig. 7.

Proof. If each I_x is a unit interval and $t_l(x) = t_r(x)$, then c_x is the center of I'_x as well as I_x, and thus $x < y$ in the Fishburn representation if and only if $c_x < c_y$ and $|I'_x \cap I'_y|$ is less than half of either interval length. □

4. Proper and unit tolerance orders

In contrast to the main theorem of this paper, in [1] there is a family of examples of proper tolerance graphs that are not unit tolerance graphs. In this section we briefly describe the order-theoretic approach to constructing those examples. The basic building block for these orders is the four-element ordered set we refer to as a 'hook', shown in Fig. 6 with the vertices numbered in the order of one possible central extension of a tolerance representation.

Lemma 3. *In any* 50% *tolerance representation of the hook in Fig.* 6 *in which the intervals have centers* $c_1 < c_2 < c_3 < c_4$, $c_4 - c_3 > c_3 - c_1$.

Proof. Since vertices 3 and 1 are incomparable, either $c_1 \in I_3$ or $c_3 \in I_1$. But $c_3 > c_2$ and c_2 is greater than all members of I_1, so $c_1 \in I_3$. Thus $c_3 - c_1$ is less than the half-length $c_3 - \mathrm{le}(3)$ of I_3. Since c_4 is greater than all members of I_3, $c_4 - c_3$ is greater than this half-length, and the conclusion follows. □

Now suppose that, as in Fig. 7, we have additional vertices 5–8 such that 3–6 form a hook and 5–8 form a hook. Note that we do not require that our ordered set be isomorphic to that of Fig. 7, for example vertex 5 could be over vertex 1 without violating the restrictions that the three sets of vertices given are hooks.

Then

$$
\begin{aligned}
c_8 - c_5 &= c_8 - c_7 + c_7 - c_5 \\
&> c_7 - c_5 + c_6 - c_5 \\
&> c_6 - c_5 + c_6 - c_5 \\
&> c_5 - c_3 + c_5 - c_3 \\
&> c_5 - c_3 + c_4 - c_3 \\
&> c_5 - c_3 + c_3 - c_1 = c_5 - c_1
\end{aligned}
$$

This is gives us

Lemma 4. *Suppose that $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8$ is a central extension of a 50% tolerance representation of an order on $\{1,2,3,4,5,6,7,8\}$ in which $\{1,2,3,4\}$, $\{3,4,5,6\}$ and $\{5,6,7,8\}$ form hooks. Then $c_8 - c_5 > c_5 - c_1$.*

Now suppose we add one more vertex x above vertex 1 as in Fig. 7(b) and seek a 50% tolerance representation in which $c_x < c_5$. Then the half length of I_x is less than $c_5 - c_1$, and therefore the right endpoint of x is less than c_8, so that $x \prec 8$. This proves the following lemma

Lemma 5. *An order on $\{1,2,3,4,5,6,7,8,x\}$ in which*
- *$\{1,2,3,4\}$, $\{3,4,5,6\}$ and $\{5,6,7,8\}$ are hooks,*
- *$x \succ 1$, and*
- *$x \not\prec 8$*

has no 50% tolerance representation in which $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8$ and $c_x < c_5$.

Of course it is not yet clear whether there is a proper representation of an ordered set with a proper tolerance representation as described in Lemma 5. However it is possible to reduce this question to a straightforward matter of linear programming. Note that vertex 2 cannot be greater than vertex 3 because of the order of their centers. If vertex 3 is incomparable with vertex 1, then it cannot be over vertex 2 either. Thus, if we can make vertices 1 and 3 incomparable, make vertex 1 less than vertex 2, and make vertices 2 and 3 less than vertex 4, we have all the comparabilities of the hook. The statement that vertex i is over vertex j can be described by inequalities by

$$\mathrm{le}(i) > \mathrm{re}(j) - t(j) \quad \text{and} \quad \mathrm{re}(j) < \mathrm{le}(i) + t(i).$$

By scaling we may assume that each strict inequality must be strict by at least one unit. So these become

$$\mathrm{le}(i) \geqslant \mathrm{re}(j) - t(j) + 1 \quad \text{and} \quad \mathrm{re}(j) \leqslant \mathrm{le}(i) + t(i) - 1.$$

The fact that the representation is proper and the statement that $c(i) > c(j)$ may be expressed in inequalities by

$$\mathrm{re}(i) \geqslant \mathrm{re}(j) + 1, \qquad \mathrm{le}(i) \geqslant \mathrm{le}(j) + 1.$$

When the center of i is above the center of j, the statement that i and j are incomparable may be expressed as one of the two inequalities

$$\mathrm{re}(j) - \mathrm{le}(i) \geqslant t(i) \quad \text{or} \quad \mathrm{re}(j) - \mathrm{le}(i) \geqslant t(j)$$

Note that in the hook on $\{1, 2,. 3, 4\}$, the incomparability of vertices 1 and 3 may only be expressed by the inequality

$$\mathrm{re}(1) - \mathrm{le}(3) \geqslant t(3),$$

because if the overlap of vertices 1 and 3 were greater than $t(1)$, so would be the overlap of vertices 1 and 2, and this is impossible since $1 \prec 2$ in the hook. This means that the only inequality we may use the express the incomparability of the first and third elements in a hook is $\mathrm{re}(i) - \mathrm{le}(j) \geqslant t(j)$ (with $j = i + 2$). Thus all the required relationships among the vertices in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ may be expressed as a system of linear inequalities. The only other relationships specified in Lemma 5 were that $x \succ 1$, $c_x < c_5$, and x is incomparable to vertex 8 (since $c_x < c_5$, $x \not\prec 8$ is equivalent to x and 8 being incomparable). The first two relationships may be expressed as inequalities as above. Since $c_x < c_5$, the overlap of I_8 and I_x must be less than that of I_8 and I_5, so the overlap of I_8 and I_x cannot be more than t_8 because then vertex 5 would not be less than vertex 8. Thus there is only one inequality that expresses the fact that vertices x and 8 are incomparable. Therefore there is no system of linear equalities which captures all the required relationships among the vertices $\{1, 2, \ldots, 8, x\}$. Using Lindo, we minimized the difference $\mathrm{re}(8) - \mathrm{le}(1)$ subject to the inequalities that describe the conditions in Lemma 5, and we obtained the following intervals and tolerances:

$$I_1 = [0, 11], \quad t_1 = 11, \qquad I_2 = [1, 12], \quad t_2 = 11, \qquad I_3 = [2, 19], \quad t_3 = 9,$$

$$I_4 = [11, 20], \quad t_4 = 9, \qquad I_5 = [13, 22], \quad t_5 = 6, \qquad I_6 = [17, 23], \quad t_6 = 6,$$

$$I_7 = [18, 24], \quad t_7 = 4, \qquad I_8 = [21, 25], \quad t_8 = 4, \qquad I_x = [12, 21], \quad t_x = 0.$$

This gives us the ordered set shown in Fig. 8(a). This does not complete the proof that our example is proper but not unit, because there could be (and is) a unit representation of this order in which the centers of the intervals do not lie in the specified order. Thus we ask, in light of Lemmas 1 and 2 if it is possible to add vertices to Fig. 8 in such a way that in any proper tolerance representation, the centers for the intervals must occur in the order we specified. The answer is that adding the two vertices y and z shown in Fig. 8(b) is sufficient. To prove this we need one more lemma, analogous to Lemmas 1 and 2, but valid only for tolerance orders.

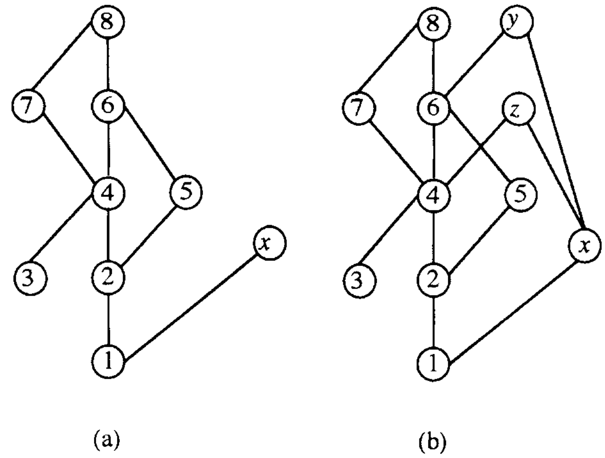

Fig. 8.

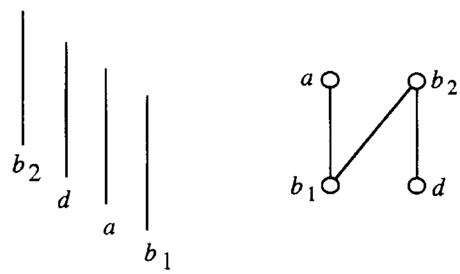

Fig. 9.

Lemma 6. *In any proper tolerance representation of the four element set $\{a, b_1, b_2, d\}$ with $a \succ b_1$, $b_1 \prec b_2$, $b_2 \succ d$, and no other comparabilities, $c_d < c_a$.*

Proof. If we assume the contrary, then as illustrated in Fig. 9, we have that $c_{b_1} < c_a < c_d < c_{b_2}$. Now since $a \succ b_1$ we have $|I_a \cap I_{b_1}| < t_a$. Now a and b_2 are incomparable, but the overlap of I_a and I_{b_2} is less than that of I_d and I_{b_2}, so $|I_a \cap I_{b_2}|$ cannot be greater than t_{b_2}. Thus $t_a < |I_a \cap I_{b_2}|$. Using once again that the overlap of I_a and I_{b_2} is less than that of I_d and I_{b_2}, gives us

$$|I_a \cap I_{b_1}| < t_a < |I_{b_2} \cap I_a| < |I_d \cap I_{b_2}|.$$

By symmetry we have

$$|I_d \cap I_{b_2}| < t_d < |I_{b_1} \cap I_d| < |I_a \cap I_{b_1}|,$$

and by transitivity, $|I_a \cap I_{b_1}| < |I_a \cap I_{b_1}|$, a contradiction. Thus c_d must be less than c_a after all. □

Theorem 3. *There is no unit tolerance representation of the order shown in Fig. 8(b).*

Proof. We show that in any central extension of a proper representation of the order, $c_i < c_{i+1}$ for i between 1 and 7, and $c_x < c_5$. First we apply the Lemma 6 to the ordered set $\{x, z, 2, 5\}$ to obtain $c_x < c_5$. Now we apply Lemma 2 to the ordered set $\{x, y, 7, 8\}$ to obtain that $c_y > c_8$. Next we apply Lemma 1 three times in succession to the ordered sets $\{6, 7, 8, y\}$, $\{4, 5, 6, 7\}$, and $\{2, 3, 4, 5\}$ to obtain that $c_6 < c_7 < c_8$, then $c_4 < c_5 < c_6$, and finally $c_2 < c_3 < c_4$. This completes the proof. □

Finally there is the question of whether adding y and z to the ordered set in Fig. 8(a) results in an ordered set with a proper tolerance representation.

Theorem 4. *There is a proper tolerance representation of the order shown in Fig. 8(b).*

Proof. To the interval representation obtained for the order of Fig. 8(a), add the interval [24, 25] for y and [22, 25] for z and give each of y and z tolerance 0. □

We would like to thank Lecretia Wilson for setting up and solving the linear programming problem described above.

5. Graph theoretic interpretation

There is a natural interpretation of Theorem 1 in terms of ordered subgraphs of ordered graphs. We adapt the definitions of ordered graphs in [6, 7, 14] as follows: An *ordered graph* $G = (V, E, L)$ is a graph (V, E) and a linear ordering L of the vertices of G. An *ordered induced subgraph* of G is an (induced) subgraph of (V, E) together with its inherited ordering. Two ordered graphs are isomorphic if there is a bijection φ between the vertex sets such that φ and φ^{-1} are order preserving functions that preserve adjacency as well.

As an example (already observed in [6, 7, 14]) of how this concept may be used, saying that $G = (V, E)$ is the incomparability graph of an ordered set with linear extension L is the same as saying that the ordered graph $G' = (V, E, L)$ has no induced 3-element ordered subgraph isomorphic to the ordered graph shown in Fig. 10. (Note that for the graph in Fig. 10 to be the complement of the comparability graph of an order with linear extension $1 < 2 < 3$, the order would have to have $1 \prec 2$, $2 \prec 3$, and $1 \not\prec 3$, an impossibility.)

How do we express the conditions of Theorem 1 in this language? Note, for example, that the conclusion of condition 2(a) of Theorem 1 can be rewritten to say that L is not the ordering $a <_L b_1 <_L b_2 <_L d$. Thus the condition forbids a certain linear extension of the vertex set of the incomparability graph of the ordered set in Fig. 2. This leads us to the following theorem in which we use the notation $L: a, b, c, d$

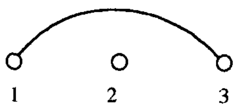

Fig. 10.

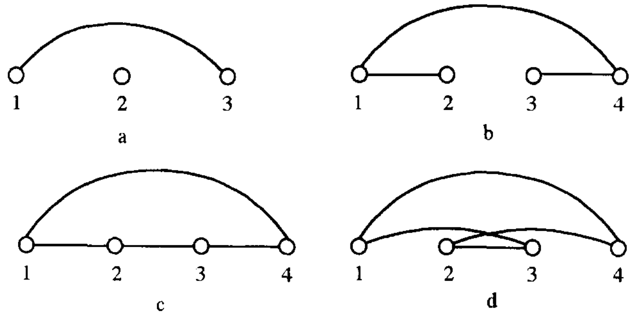

Fig. 11. The ordered graphs of Theorem 5.

to mean that L is a linearly ordered set in which a precedes b, which precedes c, which precedes d. (We suggest reading the colon as 'is given by'.)

Theorem 5. *The following statements about a graph $G = (X, E)$ are equivalent.*
1. *G is a proper bitolerance graph*
2. *There is a linear ordering $L\colon x_1, x_2, x_3, x_4$ of the vertices of G such that the ordered graph $G = (X, E, L)$ has no ordered subgraph isomorphic to any of the four ordered graphs $H = (V, F, M)$ described below and shown in Fig. 11.*
   (a) $V = \{1, 2, 3\}$, $F = \{(1, 3)\}$, M: 1, 2, 3;
   (b) $V = \{1, 2, 3, 4\}$, $F = \{(1, 2), (3, 4), (1, 4)\}$, M: 1, 2, 3, 4;
   (c) $V = \{1, 2, 3, 4\}$, $F = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$, M: 1, 2, 3, 4;
   (d) $V = \{1, 2, 3, 4\}$, $F = \{(1, 3), (2, 3) (2, 4), (1, 4)\}$, M: 1, 2, 3, 4.
3. *G is a unit bitolerance graph.*

Proof. We use the notation 'condition *a.b*' to stand for condition b of Theorem a, and the notation 'condition *a.b.c*' to stand for condition c of part b of Theorem a. We already know that G satisfies condition 5.1 (5.3) if and only if there is an orientation of the complement $\overline{G}$ that satisfies condition 1.1 (1.3). Thus it suffices to show that a graph G satisifies condition 5.2 if and only if its complement has an orientation which is an ordering satisfying condition 1.2.

We noted above that there is a linear ordering of the vertices of G satisfying condition 5.2.a if and only if $\overline{G}$ has an orientation which is an ordering of X with the given linear ordering as a linear extension. We now show that G and a linear ordering of its vertices satisfying condition 5.2.a also satisfy condition 5.2.b if and only if a corresponding orientation of $\overline{G}$ as an ordering satisfies condition 1.2.a, and that G and a linear ordering of its vertices satisfying condition 5.2.a also satisfy condition 5.2.c and 5.2.d if and only if a corresponding orientation of $\overline{G}$ satisfies condition 1.2.b.

Suppose now that G and a linear ordering of its vertices satisfy condition 5.2.a and that the ordering P is an orientation of $\overline{G}$. (Equivalently, we could assume that P is an ordering of X with linear extension L, and that G is its incomparability graph.)

Now suppose further that P satisfies condition 1.2.a. As noted above, this means that the restriction of the ordering L and $V = \{a, b_1, b_2, d\}$ is not the ordering $M: a, b_1, b_2, d$. But the edge set of the induced subgraph of G on V is $F = \{(a, b_2), (b, d), (a, d)\}$. Thus condition 1.2.a implies that the ordered graph $G = (X, E, L)$ does not have the ordered subgraph $H = (V, F, M)$. Further, any ordered subgraph of G isomorphic to F will have only orientations isomorphic to the restriction of P to V or its dual (all order relations reversed). Since H is isomorphic to the graph of condition 5.2.b, this proves that G satisfies condition 5.2.b. The converse may be proved by reversing the argument.

Now suppose instead that P satisfies condition 1.2.b. This implies that the restriction of L to the set $V = \{a_1, a_2, b_1, b_2\}$ is not $M: a_1, b_1, a_2, b_2$, $M: b_1, a_1, b_2, a_2$, $M: a_1, a_2, b_1, b_2$, or $M: b_1, b_2, a_1, a_2$. But the edge set of the induced ordered subgraph of G on the set $V = \{a_1, a_2, b_1, b_2\}$ is $F = \{(a_1, b_1), (b_1, a_2), (a_2, b_2), (a_1, b_2)\}$. The ordered graphs with $M: a_1, b_1, a_2, b_2$ and $M: b_1, a_1, b_2, a_2$ are isomorphic to the ordered graph of condition 5.2.c, and the ordered graphs with $M: a_1, a_2, b_1, b_2$ and $M: b_1, b_2, a_1, a_2$ are isomorphic to the ordered graph of condition 5.2.d. Further, any orientation of the complement of $H = (V, F, M)$ is P or an order obtained by reversing one or both of the order relations of P (which yields an ordering isomorphic to P). Again, the four forbidden linear extensions will yield ordered graphs isomorphic to those of conditions 5.2.c or 5.2.d. Thus G satisfies conditions 5.2.c and 5.2.d. The converse may be proved by reversing the argument. □

This theorem does not simplify the recognition problem for proper or unit bitolerance graphs, because as shown in [7], given an ordered graph H the decision problem of whether there is an ordering of the vertex set of a graph $G = (V, E)$ so that the resulting ordered graph does not contain H as an ordered subgraph is NP-complete for a wide variety of graphs H, including that of part 2(d) of Theorem 5. However this does not mean that the recognition problem for proper bitolerance graphs in NP-complete, because it involves recognizing graphs that exclude all the ordered graphs of Theorem 5 as ordered subgraphs. This levels us with a tantalizing problem; what is the complexity status of recognizing proper bitolerance graphs?

The recognition problem is especially interesting in light of the recent result of Shull and Trenk [20] that the directed graph concepts of proper bitolerance digraphs, unit bitolerance digraphs, and interval catch digraphs are all equivalent. Interval catch digraphs are recognizable in polynomial time [17], so this solves the directed version of the recognition problem. While a representation of a bitolerance graph yields a bitolerance digraph in a natural way, without a representation, we have as yet no way to associate with a possible bitolerance graph a digraph to test to see if it is an interval catch digraph. Thus we cannot take advantage of Shull and Trenk's result to aid in the recognition of proper bitolerance graphs.

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